# ON THE TOPOLOGY AND COMBINATORICS OF DECOMPOSABLE ARRANGEMENTS 

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#### Abstract

A complex hyperplane arrangement $\mathscr{A}$ is said to be decomposable if there are no elements in the degree 3 part of its holonomy Lie algebra besides those coming from the rank 2 flats. When this purely combinatorial condition is satisfied, it is known that the associated graded Lie algebra of the arrangement group $G$ decomposes (in degrees greater than 1) as a direct product of free Lie algebras. It follows that the $I$-adic completion of the Alexander invariant $B(G)$ also decomposes as a direct sum of "local" invariants and the Chen ranks of $G$ are the sums of the local contributions. Moreover, if $B(G)$ is separated, then the degree 1 cohomology jump loci of the complement of $\mathscr{A}$ have only local components, and the algebraic monodromy of the Milnor fibration is trivial in degree 1 .


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## 1. Introduction

1.1. Hyperplane arrangements. An arrangement of hyperplanes is a finite collection of codimension-1 linear subspaces in a finite-dimensional, complex vector space. One of the main goals of the subject is to decide whether a given topological invariant of the complement $M=M(\mathscr{A})$ is combinatorially determined, and, if so, to express it explicitly in terms of the intersection lattice $L(\mathscr{A})$.

At one extreme are the Betti numbers $b_{q}(M)$, which may be computed solely in terms of the Möbius function $\mu: L(\mathscr{A}) \rightarrow \mathbb{Z}$, and the cohomology ring $H^{*}(M ; \mathbb{Z})$, which is a graded algebra with degree-1 generators indexed by the hyperplanes, subject to relations defined in terms of $L(\mathscr{A})$. At the other extreme is the fundamental group of the complement, $G=G(\mathscr{A})$, which also admits a presentation with generators indexed by $\mathscr{A}$ and with as many commutator-relators as $b_{2}(M)$, yet is not always combinatorially determined. Other invariants fall somewhere in between. For instance, the LCS ranks and the Chen ranks of arrangement groups are combinatorially determined yet the torsion in the LCS quotients is not, while the degree 1 characteristic varieties consist of subtori that are combinatorially determined but also translated subtori whose combinatorial status is still largely unknown.

In this paper, we narrow the lens, and focus on a combinatorially-defined class of arrangements for which many of the aforementioned difficulties disappear, yet several unresolved questions still remain. Following [47], we say that an arrangement $\mathscr{A}$ is decomposable if the degree-3 part of the holonomy Lie algebra $\mathfrak{h}(\mathscr{A})$ consists only of contributions coming from the flats in $L_{2}(\mathscr{A})$. When this condition is satisfied, the associated graded Lie algebra of $G$ decomposes (in degrees greater than 1) as a direct product of free Lie algebras determined by the Möbius function of $L(\mathscr{A})$ [47], and all the nilpotent quotients of $G$ are also combinatorially determined [50].

The main part of our analysis concerns the Alexander invariant, $B(G)=G^{\prime} / G^{\prime \prime}$, viewed as a module over the group ring of $G_{\mathrm{ab}}=G / G^{\prime}$ and endowed with the topology defined by the filtration by powers of the augmentation ideal $I$. Pursuing work started in [17], we show here the following: If $\mathscr{A}$ is decomposable over $\mathbb{Q}$, then the $I$-adic completion of $B(G) \otimes \mathbb{Q}$ also decomposes as a direct sum of "local" invariants, and the Chen ranks of $G$ are the sums of the local contributions. Furthermore, we show that $B(G)$ is decomposable if and only if it separated in the $I$-adic topology and $\mathscr{A}$ is decomposable. If this is the case (upon
tensoring with $\mathbb{Q})$, then the characteristic variety $\mathscr{V}_{1}(M(\mathscr{A}))$ has only components arising from the 2 -flats of $\mathscr{A}$, and the algebraic monodromy of the Milnor fibration is trivial in degree 1.
1.2. Lie algebras and Alexander invariants. To describe in more detail our work, we start by defining the Lie algebras that come into play here. Given a group $G$, its lower central series (LCS) is defined inductively by setting $\gamma_{1}(G)=G$ and $\gamma_{k+1}(G)=\left[G, \gamma_{k}(G)\right]$ for $k \geqslant 1$. This series is both normal and central; therefore, its successive quotients, $\operatorname{gr}_{k}(G)=\gamma_{k}(G) / \gamma_{k+1}(G)$, are abelian groups. The associated graded Lie algebra, $\operatorname{gr}(G)$, is the direct sum of the groups $\operatorname{gr}_{k}(G)$, with Lie bracket (compatible with the grading) induced from the group commutator. The Chen Lie algebra, $\operatorname{gr}\left(G / G^{\prime \prime}\right)$, is simply the associated graded Lie algebra of the maximal metabelian quotient of $G$.

Assume now that $G$ is finitely generated. The LCS quotients of $G$ are then also finitely generated; we let $\phi_{k}(G):=\operatorname{rank}\left(\operatorname{gr}_{k}(G)\right)$ be the ranks of these abelian groups and we let $\theta_{k}(G):=\operatorname{rank}\left(\operatorname{gr}_{k}\left(G / G^{\prime \prime}\right)\right)$ be the Chen ranks of $G$. As shown by Massey in [41], the $I$-adic filtration on $B(G)$ coincides with the LCS-filtration on $G / G^{\prime \prime}$, after a shift of 2; therefore, $\operatorname{gr}_{k}(B(G))=\operatorname{gr}_{k+2}\left(G / G^{\prime \prime}\right)$ for all $k \geqslant 0$.

The holonomy Lie algebra, $\mathfrak{h}(G)$ is a finitely presented, quadratic Lie algebra built solely in terms of cohomological data associated to the group $G$. It is defined as the quotient of the free Lie algebra on the free abelian group $G_{\mathrm{abf}}=G_{\mathrm{ab}} /$ Tors by the Lie ideal generated by the image of the dual of the cup-product map $H^{1}(G) \wedge H^{1}(G) \rightarrow H^{2}(G)$. The Lie algebra $\mathfrak{h}(G)$ is a graded Lie algebra that maps surjectively to $\operatorname{gr}(G)$; moreover, its maximal metabelian quotient, $\mathfrak{h}(G) / \mathfrak{h}(G)^{\prime \prime}$, maps surjectively to $\operatorname{gr}\left(G / G^{\prime \prime}\right)$. Following [47], we use the holonomy Lie algebra to construct an infinitesimal version of the Alexander invariant, $\mathfrak{B}(G)=\mathfrak{h}(G)^{\prime} / \mathfrak{h}(G)^{\prime \prime}$, which is a graded module over the symmetric algebra $\operatorname{Sym}\left(G_{\text {abf }}\right)$.

If the group $G$ is 1-formal-which is the case when $G=G(\mathscr{A})$ is an arrangement group-these seemingly disparate strands tie together much more tightly, at least over the rationals. For instance, the map $\mathfrak{b}(G) \otimes \mathbb{Q} \rightarrow \operatorname{gr}(G) \otimes \mathbb{Q}$ is an isomorphism [66], and so is the map $\mathfrak{b}(G) / \mathfrak{h}(G)^{\prime \prime} \otimes \mathbb{Q} \rightarrow \operatorname{gr}\left(G / G^{\prime \prime}\right) \otimes \mathbb{Q}[47]$, while the completions of $B(G)$ and $\mathfrak{B}(G)$ are isomorphic, after tensoring with $\mathbb{Q}$ [24].
1.3. Holonomy, localization, and decomposability. For an arrangement $\mathscr{A}$, the holonomy Lie algebra $\mathfrak{h}(\mathscr{A})=\mathfrak{h}(G(\mathscr{A}))$ depends only on the truncated Orlik-Solomon algebra $H^{\leqslant 2}(M(\mathscr{A}) ; \mathbb{Z})$, and thus, only on the intersection poset $L_{\leqslant 2}(\mathscr{A})$. An explicit presentation for $\mathfrak{b}(\mathscr{A})$ was first given by Kohno in [33].

The localization of $\mathscr{A}$ at a flat $X \in L_{2}(\mathscr{A})$ is the sub-arrangement $\mathscr{A}_{X}$ consisting of those hyperplanes that contain $X$. The inclusion $\mathscr{A}_{X} \hookrightarrow \mathscr{A}$ induces a map between complements, $M(\mathscr{A}) \hookrightarrow M(\mathscr{A})$, which in turn induces a (split) surjection from $G(\mathscr{A})$ to $G\left(\mathscr{A}_{X}\right) \cong$ $F_{\mu(X)} \times \mathbb{Z}$. We obtain in this fashion an epimorphism $\mathfrak{h}(\mathscr{A}) \rightarrow \mathfrak{h}\left(\mathscr{A}_{X}\right)$. As shown in [47],
these maps assemble into a morphism of graded Lie algebras,

$$
\begin{equation*}
\mathfrak{h}(\mathscr{A}) \longrightarrow \mathfrak{h}(\mathscr{A})^{\mathrm{loc}}:=\prod_{X \in L_{2}(\mathscr{A})} \mathfrak{h}\left(\mathscr{A}_{X}\right), \tag{1.1}
\end{equation*}
$$

which is a surjection in degrees $k \geqslant 3$ and an isomorphism in degree $k=2$. The arrangement is said to be decomposable if $h_{3}(\mathscr{A}) \cong \mathfrak{h}(\mathscr{A})_{3}^{\text {loc }}$, in which case the maps $\mathfrak{h}^{\prime}(\mathscr{A}) \rightarrow \mathfrak{h}^{\prime}(\mathscr{A})^{\text {loc }}$ and $\mathfrak{h}(G) \rightarrow \operatorname{gr}(G)$ are isomorphisms, and $\mathfrak{h}(\mathscr{A})$ is torsion-free.

The decomposability property in inherited by sub-arrangements ([47]) and is preserved under taking products of arrangements (Proposition 8.2). Graphic arrangements are decomposable precisely when the corresponding graph contains no 4-cliques ([47]), while split-solvable arrangements are always decomposable (Corollary 8.4).
1.4. Alexander invariants of arrangements. Given an arrangement $\mathscr{A}$, we let $B(\mathscr{A})=$ $G(\mathscr{A})^{\prime} / G(\mathscr{A})^{\prime \prime}$ be its Alexander invariant, viewed as a module over the Laurent polynomial ring $R=\mathbb{Z}\left[G(\mathscr{A})_{\mathrm{ab}}\right]$. The structure of this module, and that of its associated graded module, $\operatorname{gr}(B(G))$, holds rich and varied information regarding the Chen ranks of $G(\mathscr{A})$, the characteristic and resonance varieties of $M(\mathscr{A})$, and the algebraic monodromy of the Milnor fibration of $\mathscr{A}$.

For each flat $X \in L_{2}(\mathscr{A})$, the homomorphism $G(\mathscr{A}) \rightarrow G\left(\mathscr{A}_{X}\right)$ induces an epimorphism $B(G(\mathscr{A})) \rightarrow B\left(G\left(\mathscr{A}_{X}\right)\right)$. Letting $B(\mathscr{A})^{\text {loc }}$ be the direct sum of all the "local" Alexander invariants $B\left(G\left(\mathscr{A}_{X}\right)\right)$, viewed as $R$-modules by restriction of scalars, we obtain a map of $R$-modules, $B(\mathscr{A}) \rightarrow B(\mathscr{A})^{\text {loc }}$.

In a similar fashion, we let $\mathfrak{B}(\mathscr{A})=\mathfrak{h}(\mathscr{A})^{\prime} / \mathfrak{h}(\mathscr{A})^{\prime \prime}$ be the infinitesimal Alexander invariant of the arrangement, viewed as a module over the polynomial ring $S=\operatorname{gr}(R)$. The Lie algebra map $\mathfrak{h}(\mathscr{A}) \rightarrow \mathfrak{h}(\mathscr{A})^{\text {loc }}$ from (1.1) induces a map of graded $S$-modules, $\mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}(\mathscr{A})^{\text {loc }}$. In Theorem 9.1, we prove the following.

Theorem 1.1. For any arrangement $\mathscr{A}$, the morphisms $\mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}(\mathscr{A})^{\mathrm{loc}}$ and $B(\mathscr{A}) \otimes$ $\mathbb{Q} \rightarrow B(\mathscr{A})^{\text {loc }} \otimes \mathbb{Q}$ are surjective.

As an application, we recover the following lower bound on the Chen ranks of arrangement groups, first established in [17] by other methods:

$$
\begin{equation*}
\theta_{k}(G(\mathscr{A})) \geqslant(k-1) \sum_{X \in L_{2}(\mathscr{A})}\binom{\mu(X)+k-2}{k} \tag{1.2}
\end{equation*}
$$

for all $k \geqslant 2$, with equality for $k=2$.
1.5. Decomposable Alexander invariants. We say that the Alexander invariant of an arrangement $\mathscr{A}$ decomposes if the canonical map $B(\mathscr{A}) \rightarrow B(\mathscr{A})^{\text {loc }}$ is an isomorphism. A similar definition was first made in $[17, \S 6.4]$ in regards to the $I$-adic completion of this
 variant decomposes if the map $\mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}(\mathscr{A})^{\text {loc }}$ is an isomorphism. In all three cases, analogous definitions work over the rationals.

A natural question arises: What is the relationship between the decomposability of $\mathscr{A}$ a purely combinatorial notion that depends only on $L_{\leqslant 2}(\mathscr{A})$-and that of $B(\mathscr{A})$-a notion that depends a priori on the topology of $M(\mathscr{A})$ ? Analogous questions may be raised about the decomposability of $B(\mathscr{A})$ over $\mathbb{Q}$, as well as the decomposability of $\widehat{B(\mathscr{A})}$ and $\mathfrak{B}(\mathscr{A})$.

As shown in Theorem 10.6, the topological decomposability notion implies the combinatorial one, that is, if $B(\mathscr{A})$ decomposes then $\mathscr{A}$ decomposes (and likewise over $\mathbb{Q}$ ). The converse, though, is much more subtle. A first step in this direction was done in [17, Thm. 7.9], where it was shown that $\widehat{B(\mathscr{A})}$ decomposes if $\mathscr{A}$ does. We give in Theorem 10.7 a completely different proof of this result, albeit over the rationals. To ascertain the decomposability of $B(\mathscr{A})$ itself, it remains to decide whether the Alexander invariant $B(\mathscr{A})$ is separated in the $I$-adic topology, or, equivalently, whether the metabelian group $G / G^{\prime \prime}$ is residually nilpotent. This separability condition holds in many examples, but it is an open question whether it holds for all decomposable arrangements.

We summarize our results in Corollary 10.9, as follows.

## Theorem 1.2. Let $\mathscr{A}$ be a hyperplane arrangement.

(1) $\mathfrak{B}(\mathscr{A})$ is decomposable (over $\mathbb{Q}$ ) if and only if $\mathscr{A}$ is decomposable (over $\mathbb{Q}$ ).
(2) $B(\mathscr{A})$ is decomposable (over $Q$ ) if and only if $\mathscr{A}$ is decomposable and $B(\mathscr{A})$ is separated (over $\mathbb{Q}$ ).

The proofs of all these implications are self-contained, except for the integral version of the reverse implication in (2), which relies on the aforementioned result from [17]. As a consequence of part (1), we show in Corollary 10.13 that the lower bound for the Chen ranks from (1.2) holds as an equality for $\mathbb{Q}$-decomposable arrangements. This recovers results from $[17,48]$ in a slightly stronger form.
1.6. Cohomology jump loci and Milnor fibrations. We conclude with an analysis of the characteristic varieties (the jump loci for homology in rank 1 local systems) and the resonance varieties (the jump loci of the Koszul complex associated to the cohomology algebra) of the complement of a decomposable arrangement.

Let $\mathscr{A}=\left\{H_{1}, \ldots, H_{n}\right\}$ be a hyperplane arrangement in $\mathbb{C}^{d+1}$. Its complement, $M=$ $M(\mathscr{A})$, is a smooth, complex, quasi-projective variety. Hence, by a general result of Arapura [1], its (degree 1) characteristic varieties, $\mathscr{V}_{s}(M)$, are finite unions of torsion-translates of algebraic subtori of the character group $\operatorname{Hom}\left(\pi_{1}(M), \mathbb{C}^{*}\right)=\left(\mathbb{C}^{*}\right)^{n}$. Since $M$ is also a formal space, its resonance varieties, $\mathscr{R}_{s}(M)$, coincide with the tangent cone at the trivial character to $\mathscr{V}_{s}(M)$, see [18, 24]. As shown in [28], the resonance varieties of $\mathscr{A}$ may
be described solely in terms of multinets on sub-arrangements of $\mathscr{A}$. In general, though, $\mathscr{V}_{1}(M)$ may contain components which do not pass through the origin, see [57, 11, 20].

For each rank-2 flat with $\mu(X)>1$, consider the linear subspace $L_{X}=\left\{x \in \mathbb{C}^{n}\right.$ : $\sum_{H_{i} \in \mathscr{A}_{X}} x_{i}=0$ and $x_{i}=0$ if $\left.H_{i} \notin \mathscr{A}_{X}\right\}$, and let $T_{X}=\exp \left(L_{X}\right) \subset\left(\mathbb{C}^{*}\right)^{n}$ be the corresponding algebraic subtorus. In Theorem 11.4, we prove the following.

Theorem 1.3. Let $\mathscr{A}$ be $a \mathbb{Q}$-decomposable arrangement. For each $s \geqslant 1$,
(1) $\mathscr{R}_{s}(M)=\bigcup_{\substack{X \in L_{2}(\mathscr{A}) \\ \mu(X)>s}} L_{X}$.
(2) If $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated, then $\mathscr{V}_{s}(M)=\bigcup_{\substack{X \in L_{2}(\mathscr{A}) \\ \mu(X)>s}} T_{X}$.

For each hyperplane $H \in \mathscr{A}$, let $f_{H}: \mathbb{C}^{d+1} \rightarrow \mathbb{C}$ be a linear form with kernel $H$. Assigning a multiplicity vector $\mathbf{m}=\left\{m_{H}\right\}_{H \in \mathscr{A}} \in \mathbb{N}^{n}$ to the hyperplanes, we obtain a polynomial map, $f_{\mathbf{m}}=\prod_{H \in \mathscr{A}} f_{H}^{m_{H}}: \mathbb{C}^{d+1} \rightarrow \mathbb{C}$, whose restriction to the complement, $f_{\mathrm{m}}: M(\mathscr{A}) \rightarrow \mathbb{C}^{*}$, is the projection map of a smooth, locally trivial bundle, known as the Milnor fibration of the multi-arrangement $(\mathscr{A}, \mathbf{m})$. Let $F_{\mathbf{m}}$ be the typical fiber and let $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ be the monodromy of the fibration. A much-studied problem is to compute the first Betti number of $F_{\mathrm{m}}$ and find the eigenvalues of the algebraic monodromy acting on $H_{1}\left(F_{\mathbf{m}} ; \mathbb{C}\right)$; see for instance [2,14, 20, 49]. As an application of the previous theorem, we prove in Theorem 12.1 the following.

Theorem 1.4. Let $\mathscr{A}$ be an arrangement of rank 3 or higher. Suppose $\mathscr{A}$ is $\mathbb{Q}$-decomposable and $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated. Then, for any choice of multiplicities $\mathbf{m}$ on $\mathscr{A}$, the algebraic monodromy of the Milnor fibration, $h_{*}: H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right) \rightarrow H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right)$, is trivial.

For an in-depth study of the topology of the Milnor fibers of hyperplane arrangements with trivial algebraic monodromy, we refer to [62].
1.7. Structure of the paper. Roughly speaking, the paper is divided into three parts. The first part covers the general theory of Alexander invariants ( $\$ 2$ ), lower central series, associated graded Lie algebras, and Chen Lie algebras (§3), concluding with holonomy Lie algebras and infinitesimal Alexander invariants (§4).

The second part deals with some basic notions regarding hyperplane arrangements. In §5 we discuss the combinatorics of an arrangement, as it relates to the topology of its complement, while in $\S 6$ and $\S 9$ we analyze in detail the holonomy Lie algebras and the Alexander invariants of arrangements, respectively.

The third part concentrates on decomposable arrangements. It starts with their basic properties (§7) and some constructions of such arrangements (§8). It then continues with arrangements whose Alexander invariants decompose (§10), and concludes with a description of their cohomology jump loci (§11) and Milnor fibrations (§12).

## 2. Alexander invariants

We start with a detailed overview of the Alexander invariant of a group, with an emphasis on its separation and functoriality properties. Along the way, we prove a number of technical results which will be needed later on.
2.1. Derived series. Let $G$ be a group. If $H$ and $K$ are subgroups of $G$, then $[H, K]$ denotes the subgroup of $G$ generated by all elements of the form $[a, b]=a b a^{-1} b^{-1}$ with $a \in$ $H$ and $b \in K$. If both $H$ and $K$ are normal subgroups, then their commutator [H,K] is again a normal subgroup; moreover, if $\alpha: G \rightarrow H$ is a homomorphism, then $\alpha([H, K]) \subseteq$ $[\alpha(H), \alpha(K)]$.

The derived series of $G$ is defined inductively by $G^{(r)}=\left[G^{(r-1)}, G^{(r-1)}\right]$, starting with $G^{(0)}=G$. In particular, $G^{(1)}=G^{\prime}$ is the derived subgroup and $G^{(2)}=G^{\prime \prime}$. The terms of these series are fully invariant subgroups; that is, if $\alpha: G \rightarrow H$ is a group homomorphism, then $\alpha\left(G^{(r)}\right) \subseteq H^{(r)}$, for all $r$. Consequently, the derived series is a normal series, i.e., $G^{(r)} \triangleleft G$, for all $r$. Moreover, since $G^{(r-1)} / G^{(r)}$ is the abelianization of $G^{(r-1)}$, all the successive quotients of the series are abelian groups.

A group $G$ is said to be solvable if its derived series of $G$ terminates in finitely many steps; that is, $G^{(\ell)}=\{1\}$ for some integer $\ell \geqslant 0$. The smallest such integer, $\ell(G)$, is then called the derived length of $G$. Clearly, $\ell(G) \leqslant 1$ if and only if $G$ is abelian, while $\ell(G) \leqslant 2$ if and only if $G$ is metabelian. The maximal solvable quotient of $G$ of length $r$ is $G / G^{(r)}$; in particular, the maximal abelian quotient is $G / G^{\prime}$ and the maximal metabelian quotient is $G / G^{\prime \prime}$.
2.2. Alexander invariant. Since the group $G_{\mathrm{ab}}=G / G^{\prime}$ is commutative, the group-ring $R=\mathbb{Z}\left[G_{\mathrm{ab}}\right]$ is also commutative. If $G_{\mathrm{ab}}$ is finitely generated, then the ring $R$ is Noetherian; if, moreover, $G_{\mathrm{ab}}$ is torsion-free, then $R$ is a Noetherian domain.

Among the successive quotients of the derives series of a group $G$, the second one plays a special role. The Alexander invariant of $G$ is the abelian group

$$
\begin{equation*}
B(G):=G^{\prime} / G^{\prime \prime} \tag{2.1}
\end{equation*}
$$

viewed as a module over the group-ring $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$; alternatively, $B(G)=\left(G^{\prime}\right)_{\mathrm{ab}}=H_{1}\left(G^{\prime} ; \mathbb{Z}\right)$. Addition in $B(G)$ is induced from multiplication in $G$ via $\left(x G^{\prime \prime}\right)+\left(y G^{\prime \prime}\right)=x y G^{\prime \prime}$ for $x, y \in$ $G^{\prime}$, while scalar multiplication is induced from conjugation in the maximal metabelian quotient, $G / G^{\prime \prime}$, via the exact sequence

$$
\begin{equation*}
1 \longrightarrow G^{\prime} / G^{\prime \prime} \longrightarrow G / G^{\prime \prime} \longrightarrow G / G^{\prime} \longrightarrow 1 \tag{2.2}
\end{equation*}
$$

That is, $g G^{\prime} \cdot x G^{\prime \prime}=g x g^{-1} G^{\prime \prime}$ for $g \in G, x \in G^{\prime}$, with the action of $G / G^{\prime}=G_{\text {ab }}$ extended $\mathbb{Z}$-linearly to the whole of $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$. This action is well-defined, since $g \in G^{\prime}$ implies $g x g^{-1} x^{-1} \in G^{\prime \prime}$, and so $g x g^{-1} G^{\prime \prime}=x G^{\prime \prime}$.

The above construction is functorial. Indeed, let $\alpha: G \rightarrow H$ be a group homomorphism. Then $\alpha$ extends linearly to a ring map, $\tilde{\alpha}: \mathbb{Z}[G] \rightarrow \mathbb{Z}[H]$. The map $\alpha$ also restricts to homomorphisms $\alpha^{\prime}: G^{\prime} \rightarrow H^{\prime}$ and $\alpha^{\prime \prime}: G^{\prime \prime} \rightarrow H^{\prime \prime}$, and thus induces homomorphisms $G / G^{\prime} \rightarrow H / H^{\prime}$ and $G^{\prime} / G^{\prime \prime} \rightarrow H^{\prime} / H^{\prime \prime}$, which we denote by $\alpha_{\mathrm{ab}}: G_{\mathrm{ab}} \rightarrow H_{\mathrm{ab}}$ and $B(\alpha): B(G) \rightarrow B(H)$, respectively. The map $B(\alpha): B(G) \rightarrow B(H)$ can then be interpreted as a morphism of modules covering the ring map $\tilde{\alpha}_{\text {ab }}: \mathbb{Z}\left[G_{\mathrm{ab}}\right] \rightarrow \mathbb{Z}\left[H_{\mathrm{ab}}\right]$; that is,

$$
\begin{equation*}
B(\alpha)(r m)=\tilde{\alpha}_{\mathrm{ab}}(r) \cdot B(\alpha)(m) \tag{2.3}
\end{equation*}
$$

for all $r \in \mathbb{Z}\left[G_{\mathrm{ab}}\right]$ and $m \in B(G)$. Clearly, if $\alpha^{\prime}: G^{\prime} \rightarrow H^{\prime}$ is surjective, then $B(\alpha): B(G) \rightarrow$ $B(H)$ is also surjective, and if $\alpha^{\prime}$ is an isomorphism, then $B(\alpha)$ is also an isomorphism. In particular, if $\alpha$ is surjective, then $\alpha^{\prime}$ is surjective, and so $B(\alpha)$ is also surjective. Nevertheless, if $\alpha$ is injective, $B(\alpha)$ need not be injective.

Remark 2.1. Given a homomorphism $\alpha: G \rightarrow H$, let $B(H)_{\alpha}$ be the $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-module obtained from $B(H)$ by restriction of scalars via the ring map $\tilde{\alpha}_{\text {ab }}: \mathbb{Z}\left[G_{\mathrm{ab}}\right] \rightarrow \mathbb{Z}\left[H_{\mathrm{ab}}\right]$. Concretely, $B(H)_{\alpha}=B(H)$ as abelian groups, with module structure given by $g \cdot m=\alpha(g) m$ for $g \in G_{\mathrm{ab}}$ and $m \in B(G)_{\alpha}$. The map $B(\alpha): B(G) \rightarrow B(H)$ can then be viewed as the composite

$$
\begin{equation*}
B(G) \longrightarrow B(H)_{\alpha} \longrightarrow B(H) \tag{2.4}
\end{equation*}
$$

where the first arrow is a $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-linear map and the second arrow is the identity map of $B(H)$, viewed as covering the ring map $\tilde{\alpha}_{a b}$.

The next lemma gives a formula expressing the Alexander invariant of a product of two groups in terms of the Alexander invariants of the factors. Another formula of this sort, involving extension of scalars instead of restriction of scalars, is given in [17, Prop. 1.8].

Lemma 2.2. Let $G=G_{1} \times G_{2}$ be a product of two groups, and let $p_{i}: G \rightarrow G_{i}$ be the projections to the factors. Then $B(G) \cong B\left(G_{1}\right)_{p_{1}} \oplus B\left(G_{2}\right)_{p_{2}}$.

Proof. Each projection map yields an epimorphism $B\left(p_{i}\right): B(G) \rightarrow B\left(G_{i}\right)$ covering the ring map $\tilde{p}_{i}: \mathbb{Z}\left[G_{\mathrm{ab}}\right] \rightarrow \mathbb{Z}\left[\left(G_{i}\right)_{\mathrm{ab}}\right]$. By the remarks above, we get $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-epimorphisms $q_{i}: B(G) \rightarrow B\left(G_{i}\right)_{p_{i}}$. Note that $G^{\prime}=G_{1}^{\prime} \times G_{2}^{\prime}$, and thus also $G^{\prime \prime}=G_{1}^{\prime \prime} \times G_{2}^{\prime \prime}$. For an element $x=\left(x_{1}, x_{2}\right) \in G^{\prime}$, the maps $q_{i}$ take the coset $x G^{\prime \prime} \in B(G)$ to $x_{i} G_{i}^{\prime \prime} \in B\left(G_{i}\right)_{p_{i}}$, where the $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-module structure on $B\left(G_{i}\right)_{p_{i}}$ is given by $g \cdot m=g_{i} m$ for $g=\left(g_{1}, g_{2}\right) \in G$. It follows that the map $q: B(G) \rightarrow B\left(G_{1}\right)_{p_{1}} \oplus B\left(G_{2}\right)_{p_{2}},\left(x_{1}, x_{2}\right) G^{\prime \prime} \mapsto\left(x_{1} G_{1}^{\prime \prime}, x_{2} G_{2}^{\prime \prime}\right)$ is an isomorphism of $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-modules, and we are done.
2.3. Topological interpretation. The Alexander invariant of a group admits the following topological interpretation. Let $X$ be a connected CW-complex with fundamental group $\pi_{1}\left(X, x_{0}\right)=G$. (Without loss of generality, we may assume $X$ has a single 0 -cell, which we
then take as the basepoint $x_{0}$.) Lifting the cell structure of $X$ to the maximal abelian cover, $q: X^{\mathrm{ab}} \rightarrow X$, we obtain an augmented chain complex of free $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-modules,

$$
\begin{equation*}
\cdots \longrightarrow C_{2}\left(X^{\mathrm{ab}} ; \mathbb{Z}\right) \xrightarrow{\partial_{2}^{\mathrm{ab}}(X)} C_{1}\left(X^{\mathrm{ab}} ; \mathbb{Z}\right) \xrightarrow{\partial_{1}^{\mathrm{ab}}(X)} C_{0}\left(X^{\mathrm{ab}} ; \mathbb{Z}\right) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{2.5}
\end{equation*}
$$

where $C_{k}\left(X^{\text {ab }} ; \mathbb{Z}\right)=C_{k}(X ; \mathbb{Z}) \otimes \mathbb{Z}\left[G_{\mathrm{ab}}\right]$ and $\varepsilon: \mathbb{Z}\left[G_{\mathrm{ab}}\right] \rightarrow \mathbb{Z}$ is the augmentation map. Since $\pi_{1}\left(X^{\mathrm{ab}}\right)=G^{\prime}$, the Alexander invariant $B(G)=\left(G^{\prime}\right)_{\mathrm{ab}}$ is isomorphic to $H_{1}\left(X^{\mathrm{ab}} ; \mathbb{Z}\right)$, the first homology group of the chain complex (2.5), with module structure induced by the action of $G_{\text {ab }}$ on $X^{\text {ab }}$ by deck transformations; equivalently, $B(G)=H_{1}\left(X ; \mathbb{Z}\left[G_{\text {ab }}\right]\right)$.

Example 2.3. Let $X=\bigvee^{n} S^{1}$ be a wedge of $n$ circles. Identify $\pi_{1}(X)$ with the free group $F_{n}=\left\langle x_{1}, \ldots, x_{n}\right\rangle$, its abelianization $\left(F_{n}\right)_{\mathrm{ab}}$ with $\mathbb{Z}^{n}$, and the group ring $\mathbb{Z}\left[\mathbb{Z}^{n}\right]$ with the ring of Laurent polynomials $R=\mathbb{Z}\left[t_{1}^{ \pm 1}, \ldots, t_{n}^{ \pm 1}\right]$. The chain complex of the universal (abelian) cover of the $n$-torus $T^{n}=K\left(\mathbb{Z}^{n}, 1\right)$ may be viewed as the Koszul complex on $t_{1}-1, \ldots, t_{n}-1$ over $R$, with differentials $\partial_{k}^{\mathrm{ab}}=\partial_{k}^{\mathrm{ab}}\left(T^{n}\right)$ given by $\partial_{k}^{\mathrm{ab}}\left(e_{J}\right)=\sum_{i=1}^{k}(-1)^{i-1}\left(t_{i}-1\right) e_{J \backslash\left\{j_{i}\right\}}$, where $e_{J}=e_{j_{1}} \wedge \cdots \wedge e_{j_{k}}$ for a $k$-tuple $J=\left\{j_{1}, \ldots, j_{k}\right\}$. The Alexander invariant of $F_{n}$, then, may be identified with the $R$-module $B\left(F_{n}\right)=\operatorname{ker}\left(\partial_{1}^{\mathrm{ab}}\right)$. From the exactness of the Koszul complex, it follows that $B\left(F_{n}\right)$ has (finite) presentation

$$
\begin{equation*}
\bigwedge^{3} \mathbb{Z}^{n} \otimes R \xrightarrow{\partial_{3}^{\mathrm{ab}}} \bigwedge^{2} \mathbb{Z}^{n} \otimes R \longrightarrow B\left(F_{n}\right) \longrightarrow 0 \tag{2.6}
\end{equation*}
$$

This example leads us to a more general result.
Lemma 2.4. If the group $G$ is finitely generated, then the Alexander invariant $B(G)$ is a finitely presented module over the Noetherian ring $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$.

Proof. Since $G$ is finitely generated (say, with $n$ generators), its abelianization $G_{\text {ab }}$ is also finitely generated (by at most $n$ elements). Therefore, the group-ring $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$ is Noetherian. Moreover, there is an epimorphism $\alpha: F_{n} \rightarrow G$, which induces an epimorphism $B(\alpha): B\left(F_{n}\right) \rightarrow B(G)$. By (2.6), the $\mathbb{Z}\left[\mathbb{Z}^{n}\right]$-module $B\left(F_{n}\right)$ is generated by $\binom{n}{2}$ elements. It follows that the Alexander invariant $B(G)$ is also finitely generated (in fact, by at most $\binom{n}{2}$ elements), and hence it is finitely presented as a module over the Noetherian ring $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$.

If the group $G$ is finitely presented, a finite presentation for the Alexander invariant may be found via the Fox calculus [29]. To start with, if $G=\left\langle x_{1}, \ldots, x_{m} \mid r_{1}, \ldots, r_{s}\right\rangle$ is a finite presentation and $K_{G}$ is the corresponding presentation 2-complex, then $\partial_{2}^{\text {ab }}\left(K_{G}\right): \mathbb{Z}\left[G_{\mathrm{ab}}\right]^{s} \rightarrow$ $\mathbb{Z}\left[G_{\mathrm{ab}}\right]^{m}$, the second boundary map in the chain complex (2.5), coincides with the Alexander matrix $\left(\mathrm{ab}\left(\partial r_{i} / \partial x_{k}\right)\right)$ of abelianized Fox derivatives of the relators. When $G_{\mathrm{ab}}$ is torsionfree, a method for finding a presentation for $B(G)$ is outlined in [41]. We illustrate this method with an example that will be needed later on.

Example 2.5. By a classical result of R. Lyndon (see [29]), if $v_{1}, \ldots, v_{n}$ are elements of the ring $\mathbb{Z}\left[\mathbb{Z}^{n}\right]=\mathbb{Z}\left[t_{1}^{ \pm 1}, \ldots, t_{n}^{ \pm 1}\right]$ which satisfy the equation $\sum_{k=1}^{n}\left(t_{k}-1\right) v_{k}=0$, then there exists an element $r \in F_{n}^{\prime}$ such that $v_{k}=\mathrm{ab}\left(\partial r / \partial x_{k}\right)$, for $1 \leqslant k \leqslant n$. Therefore, if $f=f\left(t_{1}, \ldots, t_{n}\right) \in \mathbb{Z}\left[\mathbb{Z}^{n}\right]$, then, for each $1 \leqslant i<j \leqslant n$, we may find an element $r_{i, j} \in F_{n}^{\prime}$ such that $\mathrm{ab}\left(\partial r_{i, j} / \partial x_{k}\right)$ is equal to $\left(t_{i}-1\right) f$ if $k=i$ and $\left(1-t_{j}\right) f$ if $k=j$, and is equal to 0 , otherwise. Now consider the group $G=\left\langle x_{1}, \ldots, x_{n} \mid r_{i j}(1 \leqslant i<j \leqslant n)\right\rangle$. Clearly, $G_{\mathrm{ab}}=\mathbb{Z}^{n}$. We define a chain map from the chain complex of $\left(K_{G}\right)^{\mathrm{ab}}$ to that of $\left(T^{n}\right)^{\mathrm{ab}}$, as follows:


By definition, $B(G)=\operatorname{ker}\left(\partial_{1}^{\mathrm{ab}}\right) / \operatorname{im}\left(\partial_{2}^{\mathrm{ab}}(G)\right)$. Since the Koszul complex on the bottom is exact, a diagram chase yields the presentation $B(G)=\operatorname{coker}\left(\partial_{3}^{\mathrm{ab}}+\mathrm{id} \otimes f\right)$. In particular, when $n=2$, we have that $B(G)=R /(f)$.

Finally, suppose $f: X \rightarrow Y$ is a map between connected CW-complexes; without loss of generality, we may assume $f$ is cellular and basepoint-preserving. Let $f_{\sharp}: \pi_{1}\left(X, x_{0}\right) \rightarrow$ $\pi_{1}\left(Y, y_{0}\right)$ be the induced homomorphism on fundamental groups, and let $f^{\text {ab }}: X^{\mathrm{ab}} \rightarrow Y^{\mathrm{ab}}$ be the lift to universal abelian covers. Then the morphism $B\left(f_{\sharp}\right): B\left(\pi_{1}(X)\right) \rightarrow B\left(\pi_{1}(Y)\right)$ coincides with the induced homomorphism in first homology, $f_{*}^{\text {ab }}: H_{1}\left(X^{\text {ab }} ; \mathbb{Z}\right) \rightarrow H_{1}\left(Y^{\text {ab }} ; \mathbb{Z}\right)$, and covers the ring map $\tilde{f}_{*}: \mathbb{Z}\left[H_{1}(X ; \mathbb{Z})\right] \rightarrow \mathbb{Z}\left[H_{1}(Y ; \mathbb{Z})\right]$.
2.4. The $I$-adic completion of $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$. Let $I=I\left(G_{\mathrm{ab}}\right)$ be the augmentation ideal of the group-ring $R=\mathbb{Z}\left[G_{\mathrm{ab}}\right]$, that is, the kernel of the ring map $\varepsilon: \mathbb{Z}\left[G_{\mathrm{ab}}\right] \rightarrow \mathbb{Z}$ given by $\varepsilon\left(\sum n_{g} g\right)=\sum n_{g}$. As an abelian group, $I$ is freely generated by the elements $g-1$ with $g \neq 1$, while its $k$-th power, $I^{k}$, is generated by the $k$-fold products of such elements.

The powers of the augmentation ideal form a descending, multiplicative filtration $R \supset$ $I \supset I^{2} \supset \cdots$. This filtration defines a topology on $R$, making it into a topological ring in which the ideals $I^{k}$ for a basis of open neighborhoods of 0 . Let $\widehat{R}=\lim _{k} R / I^{k}$ be the completion of $R$ with respect to the $I$-adic topology, and let $\operatorname{gr}_{I}(R)=\oplus_{n \geqslant 0} I^{k} / I^{k+1}$ be the associated graded object. Both $\widehat{R}$ and $\operatorname{gr}_{I}(R)$ acquire in a natural way a ring structure, which is compatible with the filtration by the powers of the ideal $\widehat{I}$ (the closure of $I$ in $\widehat{R}$ ), respectively, the grading. It follows that $\operatorname{gr}_{I}(R)=\operatorname{gr}_{\hat{I}}(\hat{R})$ is a graded ring. Moreover, $\widehat{R}$ is a flat $R$-module.

When endowed with the $\widehat{I}$-adic topology, $\widehat{R}$ is also a topological ring, and the canonical map to the completion, $\iota_{R}: R \rightarrow \widehat{R}$, is a morphism in this category. The injectivity of $\iota_{R}$
is equivalent to the $I$-adic topology on $R$ being Hausdorff; in turn, this is equivalent to $\bigcap_{k \geqslant 1} I^{k}=\{0\}$, that is, 0 being a closed point.

Example 2.6. If $G=\mathbb{Z}^{n}$, a choice of basis identifies the ring $R=\mathbb{Z}\left[\mathbb{Z}^{n}\right]$ with the ring of Laurent polynomials $\mathbb{Z}\left[t_{1}^{ \pm 1}, \ldots, t_{n}^{ \pm 1}\right]$, and the ideal $I=I\left(\mathbb{Z}^{n}\right)$ with the maximal ideal $\left(t_{1}-\right.$ $\left.1, \ldots, t_{n}-1\right)$. Therefore, $\widehat{R}$ may be identified with the ring of power series $\mathbb{Z}\left[\left[x_{1}, \ldots, x_{n}\right]\right]$, so that the map $\iota_{R}$ takes $t_{i}$ to $x_{i}+1$, while $\operatorname{gr}(R)$ is the polynomial ring $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$.

Lemma 2.7. If $G_{\mathrm{ab}}$ is finitely generated, then the ring $R=\mathbb{Z}\left[G_{\mathrm{ab}}\right]$ is Noetherian and the map $\iota_{R}: R \rightarrow \widehat{R}$ is injective.

Proof. If $G_{\mathrm{ab}}=\mathbb{Z}^{n}$, then both assertions are well-known. If $G_{\mathrm{ab}}=\mathbb{Z}_{r}$, then $R=\mathbb{Z}[t] /\left(t^{r}-1\right)$ and the assertions are easily verified. The general case readily follows.

Now let $\alpha: G \rightarrow H$ be a group homomorphism, let $\alpha_{\mathrm{ab}}: G_{\mathrm{ab}} \rightarrow H_{\mathrm{ab}}$ be its abelianization, and let $\tilde{\alpha}_{\mathrm{ab}}$ be its linear extension to a ring morphism from $R=\mathbb{Z}\left[G_{\mathrm{ab}}\right]$ to $S=\mathbb{Z}\left[H_{\mathrm{ab}}\right]$.

Lemma 2.8. Suppose $\alpha_{\mathrm{ab}}: G_{\mathrm{ab}} \rightarrow H_{\mathrm{ab}}$ is surjective. Then the ring maps $\tilde{\alpha}_{\mathrm{ab}}: R \rightarrow S$, $\hat{\tilde{\alpha}}_{\mathrm{ab}}: \hat{R} \rightarrow \hat{S}$, and $\operatorname{gr}\left(\tilde{\alpha}_{\mathrm{ab}}\right): \operatorname{gr}(R) \rightarrow \operatorname{gr}(S)$ are all surjective.

Proof. Since $\tilde{\alpha}_{\mathrm{ab}}\left(\sum n_{g} g\right)=\sum n_{g} \alpha_{\mathrm{ab}}(g)$, the surjectivity of $\tilde{\alpha}_{\mathrm{ab}}$ is obvious. It follows that $\tilde{\alpha}_{\mathrm{ab}}$ maps the ideal $I=I\left(G_{\mathrm{ab}}\right)$ onto the ideal $J=I\left(H_{\mathrm{ab}}\right)$. Therefore, $\tilde{\alpha}_{\mathrm{ab}}$ induces surjections $R / I^{k} \rightarrow S / J^{k}$ for all $k \geqslant 0$, which yields the surjectivity of $\hat{\tilde{\alpha}}_{\mathrm{ab}}$, and surjections $I^{k} / I^{k+1} \rightarrow$ $J^{k} / J^{k+1}$ for all $k \geqslant 0$, which proves the surjectivity of $\operatorname{gr}\left(\tilde{\alpha}_{\mathrm{ab}}\right)$.

Lemma 2.9. Suppose $H_{\mathrm{ab}}$ is finitely generated and the map $\alpha_{\mathrm{ab}}: G_{\mathrm{ab}} \rightarrow H_{\mathrm{ab}}$ is injective. Then the maps $\tilde{\alpha}_{\mathrm{ab}}: R \rightarrow S, \hat{\tilde{\alpha}}_{\mathrm{ab}}: \hat{R} \rightarrow \hat{S}$, and $\operatorname{gr}\left(\tilde{\alpha}_{\mathrm{ab}}\right): \operatorname{gr}(R) \rightarrow \operatorname{gr}(S)$ are all injective.

Proof. The injectivity of $\operatorname{gr}\left(\tilde{\alpha}_{\mathrm{ab}}\right)$ is proved in [61, Lem. 6.4]. To establish the injectivity of $\hat{\tilde{\alpha}}_{\text {ab }}$, it is enough to show that the maps $\left(\tilde{\alpha}_{\mathrm{ab}}\right)_{n}: R / I^{k} \rightarrow S / J^{k}$ are injective, for all $k \geqslant 0$. We do this by induction, starting at $k=0$, when this is obvious. The induction step follows from the injectivity of $\mathrm{gr}_{n}\left(\tilde{\alpha}_{\mathrm{ab}}\right)$, together with the Snake Lemma applied to the commuting diagram


To prove the last claim, first observe that our hypotheses imply that $G_{\mathrm{ab}}$ is finitely generated; hence, by Lemma 2.7, the ideal $\operatorname{ker}\left(\iota_{R}\right)=\bigcap_{k \geqslant 1} I^{k}$ is trivial. Therefore, since $\hat{\tilde{\alpha}}_{\text {ab }}$ is injective, we conclude that $\tilde{\alpha}_{\mathrm{ab}}$ is injective, too.
2.5. Completion and associated graded of $B(G)$. Consider now the Alexander invariant $B=B(G)$ of a group $G$, viewed as a module over the ring $R=\mathbb{Z}\left[G_{\mathrm{ab}}\right]$. The augmentation ideal $I=I\left(G_{\mathrm{ab}}\right)$ defines a descending filtration $B \supset I B \supset I^{2} B \supset \cdots$, which in turn defines the $I$-adic topology on $B$, making it into a topological module over the topological ring $R$. We let

$$
\begin{equation*}
\widehat{B}=\lim _{k} B / I^{k} B \tag{2.9}
\end{equation*}
$$

be the $I$-adic completion of the Alexander invariant, and view it as a module over $\hat{R}$. Furthermore, we let

$$
\begin{equation*}
\operatorname{gr}(B)=\bigoplus_{k \geqslant 0} I^{k} B / I^{k+1} B \tag{2.10}
\end{equation*}
$$

be the associated graded of the Alexander invariant, viewed as a (graded) module over the ring $\operatorname{gr}(R)$. Note that $\operatorname{gr}(B)$ is generated as a $\operatorname{gr}(R)$-module by its degree 0 piece, $\operatorname{gr}_{0}(B)=B / I B$.

If the group $G$ is finitely generated, then, by Lemma 2.4, the Alexander invariant $B=$ $B(G)$ is a finitely presented module over the Noetherian ring $R$. Therefore, the canonical map to the completion, $\iota_{B}: B \rightarrow \widehat{B}$, induces an isomorphism $B \otimes_{R} \widehat{R} \rightarrow \widehat{B}$, see [5, Prop. 3.13]. Moreover, by the exactness property of completion (see e.g. [5, Prop. 10.12]), the following holds: If $R^{m} \xrightarrow{\varphi} R^{n} \rightarrow B \rightarrow 0$ is a finite presentation for $B$, then $\widehat{B}$ has presentation $\widehat{R}^{m} \xrightarrow{\hat{\varphi}} \widehat{R}^{n} \rightarrow B \rightarrow 0$.
2.6. Separated Alexander invariants. Observe that the kernel of the morphism $\iota_{B}: B \rightarrow$ $\widehat{B}$ is equal to $\bigcap_{k \geqslant 1} I^{k} B$, the closure of $\{0\}$ in the $I$-adic topology on $B$. By Krull's Theorem, $\operatorname{ker}\left(\iota_{B}\right)$ consists of all the elements of $B$ that are annihilated by $1+I$, see [5, Thm. 10.17].

We say that the Alexander invariant $B=B(G)$ is (I-adically) separated if the morphism $\iota_{B}$ is injective, that is, the topological module $B$ is a Hausdorff space. Here is an example where this happens.

Example 2.10. Let $F_{n}$ be the free group of rank $n$. As in Example 2.6, we identify the ring $R=\mathbb{Z}\left[\mathbb{Z}^{n}\right]$ with $\mathbb{Z}\left[t_{1}^{ \pm 1}, \ldots, t_{n}^{ \pm 1}\right]$ and $\widehat{R}$ with $\mathbb{Z}\left[\left[x_{1}, \ldots x_{n}\right]\right]$, so that $\iota_{R}\left(t_{i}\right)=x_{i}+1$. From Example 2.3, we know that the Alexander invariant $B=B\left(F_{n}\right)$ is the kernel of the $R$-linear map $\partial_{1}^{\text {ab }}: R^{n} \rightarrow R$ given by $\partial_{1}^{\text {ab }}\left(e_{i}\right)=t_{i}-1$. By exactness of completion, the $\hat{R}$-module $\widehat{B}$ is equal to $\operatorname{ker}\left(\hat{\partial}_{1}^{\mathrm{ab}}\right)$, where $\hat{\partial}_{1}^{\mathrm{ab}}\left(e_{i}\right)=x_{i}$. Since the map $\iota_{R}^{n}: R^{n} \rightarrow \widehat{R}^{n}$ is injective (see Lemma 2.7), it follows that its restriction to the Alexander invariant, $\iota_{B}: B \rightarrow \widehat{B}$, is also injective.

In general, though, the Alexander invariant of a finitely generated group (even a commu-tator-relators group) is not separated. In fact, as the next example shows, $\operatorname{ker}\left(\iota_{B}\right)$ may be equal to $B$, even when $B \neq 0$.

Example 2.11. Let $f=f\left(t_{1}, t_{2}\right)$ be a Laurent polynomial in $R=\mathbb{Z}\left[\mathbb{Z}^{2}\right]$ and let $G=$ $\left\langle x_{1}, x_{2} \mid r\right\rangle$ be the corresponding commutator-relator group, constructed in Example 2.5, so that $B=R /(f)$. Assume that $\varepsilon(f)=1$, yet $f$ is not a monomial in $t_{1}, t_{2}$ (for instance, take $\left.f=2-t_{1}\right)$. Then $f-1$ belongs to the ideal $I=I\left(\mathbb{Z}^{2}\right)$ and $B \neq 0$. On the other hand, $f$ is invertible in $\widehat{R}$, and so $\widehat{B}=\widehat{R} /(f)=0$; alternatively, note that $I B=B$, and so $\operatorname{ker}\left(\iota_{B}\right)=\bigcap_{k \geqslant 1} I^{k} B=B$.

As the next lemma shows, the notion of separability behaves well with respect to (finite) direct products of groups.

Lemma 2.12. Let $G=G_{1} \times G_{2}$ be the product of two groups. Then $B(G)$ is separated if and only if both $B\left(G_{1}\right)$ and $B\left(G_{2}\right)$ are separated.

Proof. By Lemma 2.2, the $\mathbb{Z}\left[G_{\text {ab }}\right]$-module $B(G)$ is isomorphic to $B\left(G_{1}\right)_{p_{1}} \oplus B\left(G_{2}\right)_{p_{2}}$, where $p_{i}: G \rightarrow G_{i}$ are the projections maps. Observe that $I^{k} B\left(G_{i}\right)_{p_{i}}=\left(I_{i}\right)^{k} B\left(G_{i}\right)$ for all $k \geqslant 1$, where $I$ and $I_{i}$ are the augmentation ideals of $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$ and $\mathbb{Z}\left[\left(G_{i}\right)_{\mathrm{ab}}\right]$, respectively. Therefore, $B\left(G_{i}\right)_{p_{i}}$ is separated if and and only if $B\left(G_{i}\right)$ is separated. The claim now follows from the fact that completion commutes with direct sums.

From Example 2.10 and Lemma 2.12, we obtain the following immediate corollary.
Corollary 2.13. Let $G=F_{n_{1}} \times \cdots \times F_{n_{r}}$ be a finite direct product of finitely generated free groups. Then $B(G)$ is separated.
2.7. Naturality properties. We conclude this section with a description of the functoriality properties of the aforementioned constructions with Alexander invariants.

Recall from Section 2.2 that a homomorphism $\alpha: G \rightarrow H$ induces a map between Alexander invariants, $B(\alpha): B(G) \rightarrow B(H)$, which covers the ring map $\tilde{\alpha}_{\text {ab }}: R \rightarrow S$, where $R=\mathbb{Z}\left[G_{\mathrm{ab}}\right]$ and $S=\mathbb{Z}\left[H_{\mathrm{ab}}\right]$. Clearly, $B(\alpha)$ is a morphism of topological modules, that is, it sends $I^{k} B(G)$ to $J^{k} B(H)$ for all $k \geqslant 1$, where $I=I\left(G_{\mathrm{ab}}\right)$ and $J=I\left(H_{\mathrm{ab}}\right)$. Moreover, $B(\alpha)$ factors as the composite $B(G) \rightarrow B(H)_{\alpha} \rightarrow B(H)$, where $B(H)_{\alpha}$ is the $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-module obtained from $B(H)$ by restriction of scalars via the ring map $\tilde{\alpha}_{\mathrm{ab}}: \mathbb{Z}\left[G_{\mathrm{ab}}\right] \rightarrow \mathbb{Z}\left[H_{\mathrm{ab}}\right]$.

Lemma 2.14. If the $\mathbb{Z}\left[H_{\mathrm{ab}}\right]$-module $B(H)$ is separated, then the induced $\mathbb{Z}\left[G_{\mathrm{ab}}\right]$-module $B(H)_{\alpha}$ is also separated.

Proof. By definition, $B(H)_{\alpha}=B(H)$, with module structure given by $g \cdot m=\alpha(g) m$ for $g \in G_{\mathrm{ab}}$ and $m \in B(G)_{\alpha}$. It follows that $I^{k} B(H)_{\alpha} \subseteq J^{k} B(H)$ for all $k \geqslant 1$. Therefore, if $\bigcap_{k \geqslant 1} J^{k} B(H)=0$, then $\bigcap_{k \geqslant 1} I^{k} B(H)_{\alpha}=0$, and this proves the claim.

Since the map $B(\alpha): B(G) \rightarrow B(H)$ is filtration-preserving, it induces a morphism between the respective completions, $\widehat{B}(\alpha): \widehat{B(G)} \rightarrow \widehat{B(H)}$. This morphism covers the ring
map $\hat{\tilde{\alpha}}_{\mathrm{ab}}: \widehat{R} \rightarrow \widehat{S}$ and fits into the commuting diagram


Passing to associated graded objects, we obtain a morphism, $\operatorname{gr}(B(\alpha)): \operatorname{gr}(B(G)) \rightarrow$ $\operatorname{gr}(B(H))$, which covers the ring map $\operatorname{gr}\left(\tilde{\alpha}_{\mathrm{ab}}\right): \operatorname{gr}(R) \rightarrow \operatorname{gr}(S)$. Note that $\operatorname{gr}_{0}\left(\tilde{\alpha}_{\mathrm{ab}}\right): R / I \rightarrow$ $S / J$ may be identified with id: $\mathbb{Z} \rightarrow \mathbb{Z}$, while $\operatorname{gr}_{1}\left(\tilde{\alpha}_{\mathrm{ab}}\right): I / I^{2} \rightarrow J / J^{2}$ may be identified with $\alpha_{\mathrm{ab}}: G_{\mathrm{ab}} \rightarrow H_{\mathrm{ab}}$.

Lemma 2.15. Assume the groups $G$ and $H$ are finitely generated. For a homomorphism $\alpha: G \rightarrow H$, the following conditions are equivalent.
(1) The map $B(\alpha): B(G) \rightarrow B(H)$ is surjective.
(2) The map $\widehat{B(\alpha)}: \widehat{B(G)} \rightarrow \widehat{B(H)}$ is surjective.
(3) The map $\operatorname{gr}(B(\alpha)): \operatorname{gr}(B(G)) \rightarrow \operatorname{gr}(B(H))$ is surjective.

Proof. By Lemma 2.4, the Alexander invariants $B(G)$ and $B(H)$ are finitely generated modules over the Noetherian rings $\mathbb{Z}\left[G_{a b}\right]$ and $\mathbb{Z}\left[H_{a b}\right]$, respectively. Implication (1) $\Rightarrow$ (2) follows from the exactness property of completion, while (2) $\Rightarrow$ (3) follows from the fact that $\operatorname{gr}(B(H))$ is generated in degree 0 , and (3) $\Rightarrow$ (1) follows from [5, Lemma 10.23].

Lemma 2.16. Let $\alpha: G \rightarrow H$ be a homomorphism between finitely generated groups. Then,
(1) If $B(\alpha): B(G) \rightarrow B(H)$ is injective, then $\widehat{B(\alpha)}: \widehat{B(G)} \rightarrow \widehat{B(H)}$ is also injective.
(2) The map $\widehat{B(\alpha)}: \widehat{B(G)} \rightarrow \widehat{B(H)}$ is injective if and only if the map $\operatorname{gr}(B(\alpha)): \operatorname{gr}(B(G)) \rightarrow$ $\operatorname{gr}(B(H))$ is injective.
(3) If $B(G)$ is separated and $\widehat{B(\alpha)}$ is injective, then $B(\alpha)$ is injective.

Proof. Claim (1) follows again from the exactness property of completion. The forward implication in claim (2) follows from the Snake Lemma, applied to the analogue of diagram (2.8) for the module maps $\operatorname{gr}_{k}(B(\alpha))$ and $B(\alpha)_{k}: B(G) / I^{k} \rightarrow B(H) / J^{k}$, while the other implication follows from [5, Lemma 10.23]. Finally, claim (3) follows from the commutativity of diagram (2.11) and the assumption that both $\iota_{B(G)}$ and $\widehat{B(\alpha)}$ are injective.

## 3. Lower central series and Chen groups

In this section, we review the lower central series quotients and the Chen groups of a group $G$, and how the latter relate to the Alexander invariant of $G$.
3.1. Lower central series. The lower central series (LCS) of a group $G$ is defined recursively by $\gamma_{k+1}(G)=\left[G, \gamma_{k}(G)\right]$, starting with $\gamma_{1}(G)=G$. An inductive argument shows that $\left[\gamma_{k}(G), \gamma_{\ell}(G)\right] \subseteq \gamma_{k+\ell}(G)$, for all $k, \ell \geqslant 1$; in particular, the LCS is a central series, i.e., $\left[G, \gamma_{k}(G)\right] \subseteq \gamma_{k+1}(G)$ for all $k$. Moreover, its terms are fully invariant (and thus, normal) subgroups of $G$.

The successive quotients of the series, $\gamma_{k}(G) / \gamma_{k+1}(G)$, are abelian groups. The first such quotient, $G / \gamma_{2}(G)$, coincides with the abelianization $G_{\text {ab }}=H_{1}(G ; \mathbb{Z})$. If $\gamma_{k}(G) \neq 1$ but $\gamma_{k+1}(G)=1$, then $G$ is said to be a $k$-step nilpotent group. For each $k \geqslant 1$, there is a canonical projection, $\psi_{k}: G \rightarrow G / \gamma_{k+1}(G)$, where the factor group is the maximal $k$-step nilpotent quotient of $G$.

Note that $G^{(k-1)} \subseteq \gamma_{k}(G)$, with equality for $k=1$ and 2 . Consequently, every nilpotent group is solvable. As another consequence of this observation, the Alexander invariant $B(G)=G^{\prime} / G^{\prime \prime}$ surjects onto the quotient group $\gamma_{2}(G) / \gamma_{3}(G)$.

The group $G$ is called residually nilpotent if every non-identity element $g \in G$ is detected in a nilpotent quotient of $G$; that is, there is a surjective homomorphism $\varphi: G \rightarrow N$ such that $N$ is nilpotent and $\varphi(g) \neq 1$. Clearly, such an homomorphism factors through the projection $\psi_{k}: G \rightarrow G / \gamma_{k+1}(G)$, for some $k \geqslant 1$. Thus, $G$ is residually nilpotent if and only if the intersection of its lower central series, $\gamma_{\omega}(G):=\bigcap_{k \geqslant 1} \gamma_{k}(G)$, is the trivial subgroup. The finitely generated free groups $F_{n}$ and the pure braid groups $P_{n}$ are well-known examples of residually (torsion-free) nilpotent groups.
3.2. Associated graded Lie algebra. The associated graded Lie algebra of $G$ is the direct sum of the successive quotients of the lower central series. The addition in $\operatorname{gr}(G)$ is induced from the group multiplication, while the Lie bracket is induced from the group commutator. The graded pieces are the abelian groups $\operatorname{gr}_{k}(G)=\gamma_{k}(G) / \gamma_{k+1}(G)$, while the Lie bracket is compatible with the grading.

By construction, the Lie algebra $\operatorname{gr}(G)$ is generated by its degree 1 piece, $\operatorname{gr}_{1}(G)=G_{\mathrm{ab}}$; thus, if $G_{\mathrm{ab}}$ is finitely generated, then so are the LCS quotients of $G$. Likewise, the $\mathbb{Q}$-Lie algebra $\operatorname{gr}(G) \otimes \mathbb{Q}$ is generated in degree 1 by the $\mathbb{Q}$-vector space $G_{\text {ab }} \otimes \mathbb{Q}=H_{1}(G ; \mathbb{Q})$. Assume now that the first Betti number, $b_{1}(G)=\operatorname{dim}_{\mathbb{Q}} H_{1}(G ; \mathbb{Q})$, is finite; we may then define the LCS ranks of $G$ as

$$
\begin{equation*}
\phi_{k}(G):=\operatorname{dim}_{\mathbb{Q}} \operatorname{gr}_{k}(G) \otimes \mathbb{Q} . \tag{3.1}
\end{equation*}
$$

If $\alpha: G \rightarrow H$ is a group homomorphism, then $\alpha\left(\gamma_{k}(G)\right) \subseteq \gamma_{k}(H)$, and thus $\alpha$ induces a map $\operatorname{gr}(\alpha): \operatorname{gr}(G) \rightarrow \operatorname{gr}(H)$. It is readily seen that this map preserves Lie brackets and that the assignment $\alpha \leadsto \operatorname{gr}(\alpha)$ is functorial. Moreover, if $G$ and $H$ are two groups, it is readily seen that $\gamma_{k}(G \times H) \cong \gamma_{k}(G) \times \gamma_{k}(H)$ for all $k \geqslant 1$, from which it follows that $\operatorname{gr}(G \times H) \cong \operatorname{gr}(G) \times \operatorname{gr}(H)$, as graded Lie algebras.

For each $k \geqslant 1$, the canonical projection $G \rightarrow G / \gamma_{k}(G)$ induces an epimorphism $\operatorname{gr}(G) \rightarrow \operatorname{gr}\left(G / \gamma_{k}(G)\right)$, which is an isomorphism in degrees $j<k$. Moreover, as shown in [64, Lem. 6.4], the following holds.

Lemma 3.1 ([64]). For each $r \geqslant 2$, the quotient map, $G \rightarrow G / G^{(r)}$, induces an epimorphism, $\operatorname{gr}_{k}(G) \rightarrow \operatorname{gr}_{k}\left(G / G^{(r)}\right)$, which is an isomorphism for $k \leqslant 2^{r}-1$.

In the case when $r=2$, originally studied by K.-T. Chen in [7], the corresponding Lie algebra, $\operatorname{gr}\left(G / G^{\prime \prime}\right)$, is called the Chen Lie algebra of $G$. Assuming $b_{1}(G)<\infty$, we may define the Chen ranks of $G$ as

$$
\begin{equation*}
\theta_{k}(G):=\phi_{k}\left(G / G^{\prime \prime}\right)=\operatorname{dim}_{\mathbb{Q}} \operatorname{gr}_{k}\left(G / G^{\prime \prime}\right) \otimes \mathbb{Q} . \tag{3.2}
\end{equation*}
$$

In view of the above discussion, we have that $\theta_{k}(G) \leqslant \phi_{k}(G)$, with equality for $k \leqslant 3$.
Example 3.2. Let $F_{n}$ be the free group of rank $n$. The associated graded Lie algebra $\operatorname{gr}\left(F_{n}\right)$ is isomorphic to the free Lie algebra $\operatorname{Lie}\left(\mathbb{Z}^{n}\right)$. Moreover, all its graded pieces are free abelian groups, with ranks given by Witt's formula,

$$
\begin{equation*}
\phi_{k}\left(F_{n}\right)=\frac{1}{k} \sum_{d \mid k} \mu(d) n^{k / d} \tag{3.3}
\end{equation*}
$$

Equivalently, these LCS ranks can be read off from the equality $\prod_{k=1}^{\infty}\left(1-t^{k}\right)^{\phi_{k}}=1-n t$. The Chen groups $\operatorname{gr}_{k}\left(F_{n} / F_{n}^{\prime \prime}\right)$ are also free abelian, of ranks given by $\theta_{1}\left(F_{n}\right)=n$ and

$$
\begin{equation*}
\theta_{k}\left(F_{n}\right)=(k-1)\binom{n+k-2}{k} \tag{3.4}
\end{equation*}
$$

for $k \geqslant 2$, see [7].
3.3. Massey's correspondence. In [41], Massey established a simple, yet very fruitful connection between the Alexander invariant of a group $G$ and the lower central series of its maximal metabelian quotient, $G / G^{\prime \prime}$. For a full account-with complete proofs-of Massey's correspondence, we refer to [61, §6].

Let $1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1$ be an extension of groups. Choosing a set-theoretic section $\sigma: Q \rightarrow G$ of the projection map $G \rightarrow Q$ defines a function $\varphi: Q \rightarrow \operatorname{Aut}(K)$ by setting $\varphi(x)(a)=\sigma(x) a \sigma(x)^{-1}$ for $x \in Q$ and $a \in K$. Now suppose $K$ is abelian; then the map $\varphi$ is a well-defined homomorphism that puts the structure of a $\mathbb{Z}[Q]$-module on $K$. Define a filtration $\left\{K_{n}\right\}_{n \geqslant 0}$ on $K$ inductively, by setting $K_{0}=K$ and $K_{n+1}=\left[G, K_{n}\right]$. Then $K_{n}=I^{n} K$ for all $n \geqslant 0$, where $I=I(Q)$ is the augmentation ideal of $\mathbb{Z}[Q]$. Applying these observations to the exact sequence (2.2) yields the following result.

Theorem 3.3 ([41]). Let $G$ be a group, and let $I=I\left(G_{\mathrm{ab}}\right)$. Then $I^{k} B(G)=\gamma_{k+2}\left(G / G^{\prime \prime}\right)$, for all $k \geqslant 0$.

Passing to the associated graded objects, the theorem yields the following corollary.

Corollary 3.4 ([41]). For any group $G$, there are natural isomorphisms $\operatorname{gr}_{k}(B(G)) \cong$ $\operatorname{gr}_{k+2}\left(G / G^{\prime \prime}\right)$, for all $k \geqslant 0$.

Using now Lemma 3.1, we obtain isomorphisms

$$
\begin{equation*}
\operatorname{gr}_{0}(B(G)) \cong \operatorname{gr}_{2}(G) \quad \text { and } \quad \operatorname{gr}_{1}(B(G)) \cong \operatorname{gr}_{3}(G) \tag{3.5}
\end{equation*}
$$

As another immediate consequence of Theorem 3.3, we obtain the following purely group-theoretical characterization of separability for the Alexander invariant of a group.

Corollary 3.5. The Alexander invariant $B(G)$ is separated (in the I-adic topology) if and only if the group $G / G^{\prime \prime}$ is residually nilpotent.

Now suppose that $b_{1}(G)<\infty$. Then $\operatorname{gr}(B(G) \otimes \mathbb{Q})$ is a finitely generated graded module over the graded ring $\operatorname{gr}\left(\mathbb{Q}\left[G_{\mathrm{ab}}\right]\right)$. Let $\theta_{k}(G)=\operatorname{dim}_{\mathbb{Q}} \operatorname{gr}_{k}\left(G / G^{\prime \prime}\right) \otimes \mathbb{Q}$ be the Chen ranks of $G$, starting with $\theta_{1}(G)=b_{1}(G)$. As a consequence of Corollary 3.4, the Hilbert series of the rationalization of $\operatorname{gr}(B(G))$ determines the Chen ranks of $G$, as follows,

$$
\begin{equation*}
\operatorname{Hilb}(\operatorname{gr}(B(G) \otimes \mathbb{Q}), t)=\sum_{k \geqslant 0} \theta_{k+2}(G) t^{k} \tag{3.6}
\end{equation*}
$$

Example 3.6. Let $F_{n}$ be the free group of rank $n \geqslant 2$. As we saw in Example 2.3, the Alexander invariant of $F_{n}$, viewed as a module over the ring $R=\mathbb{Z}\left[\mathbb{Z}^{n}\right]$, has presentation $B\left(F_{n}\right)=\operatorname{coker}\left(\partial_{3}^{\text {ab }}\right)$. Consider now the associated graded object, $\operatorname{gr}\left(B\left(F_{n}\right)\right)$, viewed as a module over the polynomial ring $S=\operatorname{gr}(R)$. A standard Gröbner basis argument shows that $\operatorname{gr}\left(B\left(F_{n}\right)\right)$ is the cokernel of the third differential in the Koszul complex $\left(\bigwedge^{*} \mathbb{Z}^{n} \otimes S, \partial\right)$. Using the exactness of this chain complex of free $S$-modules, we find that

$$
\begin{equation*}
\operatorname{Hilb}\left(\operatorname{gr}\left(B\left(F_{n}\right)\right), t\right)=1-\frac{1-n t}{(1-t)^{n}} \tag{3.7}
\end{equation*}
$$

Applying now formula (3.6) recovers the computation of the Chen ranks of the free group $F_{n}$ from (3.4).

## 4. Holonomy Lie algebras and infinitesimal Alexander invariants

We now review two other objects associated to a group $G$ : the holonomy Lie algebra $\mathfrak{h}(G)$ and the infinitesimal Alexander invariant $\mathfrak{B}(G)$.
4.1. The holonomy Lie algebra of a group. Let $G$ be a group such that the maximal torsion-free abelian quotient $G_{\mathrm{abf}}:=G_{\mathrm{ab}} /$ Tors is finitely generated. Let $A^{*}=H^{*}(G ; \mathbb{Z})$ be the cohomology ring of $G$, and consider the map $A^{1} \otimes A^{1} \rightarrow A^{2}, a \otimes b \mapsto a \cup b$. Since the cup-product in cohomology is graded-commutative, this map factors through a homomorphism $\cup_{G}: A^{1} \wedge A^{1} \rightarrow A^{2}$. Due to our finite generation assumption on $G_{\text {abf }}$, we
have isomorphisms $\left(A^{1}\right)^{\vee} \cong G_{\text {abf }}$ and $\left(A^{1} \wedge A^{1}\right)^{\vee} \cong G_{\text {abf }} \wedge G_{\text {abf }}$, where ()$^{\vee}$ signifies $\mathbb{Z}$-dual. Dualizing the map $\cup_{G}$, we obtain the comultiplication map,

$$
\begin{equation*}
\nabla_{G}=\left(\cup_{G}\right)^{\vee}: H^{2}(G ; \mathbb{Z})^{\vee} \longrightarrow G_{\mathrm{abf}} \wedge G_{\mathrm{abf}} . \tag{4.1}
\end{equation*}
$$

In the case when $G_{\text {ab }}$ is torsion-free (and finitely generated), we may identify the $\mathbb{Z}$-dual of $H^{2}(G ; \mathbb{Z})$ with $H_{2}(G ; \mathbb{Z})$. Moreover, if we let ab: $G \rightarrow G_{\mathrm{ab}}$ be the abelianization map, then $\nabla_{G}$ may be identified with the induced homomorphism ab ${ }_{*}: H_{2}(G ; \mathbb{Z}) \rightarrow H_{2}\left(G_{\text {ab }} ; \mathbb{Z}\right)$.

Returning to the general case, let $\operatorname{Lie}\left(G_{\mathrm{abf}}\right)$ be the free $\mathbb{Z}$-Lie algebra on the free $\mathbb{Z}$ module $G_{\text {abf }}$. This is a graded Lie algebra, with grading given by bracket length; we denote by $\operatorname{Lie}_{k}\left(G_{\text {abf }}\right)$ its degree $k$ piece. Furthermore, we identify $\operatorname{Lie}_{1}\left(G_{\text {abf }}\right)=G_{\text {abf }}$ and $\operatorname{Lie}_{2}\left(G_{\text {abf }}\right)=G_{\text {abf }} \wedge G_{\text {abf }}$ via $[x, y] \mapsto x \wedge y$. With these notations and identifications, we define $\mathfrak{h}(G)$, the holonomy Lie algebra of $G$, as the quotient of the free Lie algebra on $G_{\text {abf }}$ by the Lie ideal generated by the image of the comultiplication map,

$$
\begin{equation*}
\mathfrak{h}(G):=\operatorname{Lie}\left(G_{\text {abf }}\right) / \operatorname{ideal}\left(\operatorname{im}\left(\nabla_{G}\right)\right) . \tag{4.2}
\end{equation*}
$$

Note that the ideal generated by the image of $\nabla_{G}$ is a homogeneous ideal—in fact, a quadratic ideal. Thus, the holonomy Lie algebra inherits a grading from the free Lie algebra, and this grading is compatible with the Lie bracket. In fact, $\mathfrak{h}(G)=\oplus_{k \geqslant 1} \mathfrak{b}_{k}(G)$ is a quadratic Lie algebra: it is generated in degree 1 by $\mathfrak{h}_{1}(G)=G_{\text {abf }}$, and all the relations are in degree 2. As noted in [64, Prop. 6.2], the projection maps $\psi_{k}: G \rightarrow G / \gamma_{k}(G)$ induce isomorphisms $\mathfrak{h}\left(\psi_{k}\right): \mathfrak{h}(G) \xrightarrow{\simeq} \mathfrak{h}\left(G / \gamma_{k}(G)\right)$ for all $k \geqslant 3$. In particular, the holonomy Lie algebra of $G$ depends only on its second nilpotent quotient, $G / \gamma_{3}(G)$.

The derived series of the Lie algebra $\mathfrak{h}=\mathfrak{h}(G)$ is defined inductively by setting $\mathfrak{h}^{(r)}=$ $\left[\mathfrak{h}^{(r-1)}, \mathfrak{h}^{(r-1)}\right]$, starting with $\mathfrak{h}^{(0)}=\mathfrak{h}$. In particular, $\mathfrak{h}^{(1)}=\mathfrak{h}$, is the derived Lie subalgebra and $\mathfrak{h}^{(2)}=\mathfrak{h}^{\prime \prime}$. The terms of the derived series are Lie ideals which are preserved by Lie algebra maps. Moreover, since $\mathfrak{b}$ is a graded Lie algebra, we have that $\mathfrak{h}^{\prime}=\oplus_{k \geqslant 2} \mathfrak{h}_{k}$.

The above construction is functorial. Indeed, let $\alpha: G \rightarrow H$ be a homomorphism between two groups as above; then the induced homomorphism $\alpha_{\text {abf }}: G_{\text {abf }} \rightarrow H_{\text {abf }}$ extends to a morphism $\operatorname{Lie}\left(\alpha_{\text {abf }}\right): \operatorname{Lie}\left(G_{\text {abf }}\right) \rightarrow \operatorname{Lie}\left(H_{\text {abf }}\right)$ between the respective free Lie algebras. The map $\alpha$ also induces a ring map, $\alpha^{*}: H^{*}(H ; \mathbb{Z}) \rightarrow H^{*}(G ; \mathbb{Z})$; passing to duals, it follows that $\operatorname{Lie}\left(\alpha_{\mathrm{abf}}\right)$ sends $\operatorname{im}\left(\nabla_{G}\right)$ to $\operatorname{im}\left(\nabla_{H}\right)$. Consequently, $\operatorname{Lie}\left(\alpha_{\mathrm{abf}}\right)$ induces a morphism of graded Lie algebras, $\mathfrak{h}(\alpha): \mathfrak{h}(G) \rightarrow \mathfrak{h}(H)$. It is now readily verified that $\mathfrak{h}(\beta \circ \alpha)=\mathfrak{h}(\beta) \circ \mathfrak{h}(\alpha)$. Moreover, if $\alpha$ is surjective, then $\mathfrak{h}(\alpha)$ is also surjective.

As shown in the next lemma, the construction also works well with direct products.
Lemma 4.1. Let $G$ and $H$ be two groups with $G_{\text {abf }}$ and $H_{\mathrm{abf}}$ finitely generated. There is then an isomorphism of graded Lie algebras,

$$
\mathfrak{h}(G \times H) \cong \mathfrak{h}(G) \times \mathfrak{h}(H) .
$$

Proof. Using the Künneth formula to compute cup products in $H^{\leqslant 2}(G \times H ; \mathbb{Z})$, the claim follows in a straightforward manner from the definition of the holonomy Lie algebra.

In a completely analogous fashion, one may define the holonomy Lie algebra $\mathfrak{h}(A)$ of a graded, graded-commutative algebra $A$ over a field $\mathbb{k}$, provided that $A^{0}=\mathbb{k}$ and $A^{1}$ is finite-dimensional, by letting $\nabla_{A}: A_{2} \rightarrow A_{1} \wedge A_{1}$ be the $\mathbb{k}$-dual of the multiplication map $A^{1} \wedge A^{1} \rightarrow A^{2}$ and setting $\mathfrak{h}(A)=\operatorname{Lie}\left(A_{1}\right) / \operatorname{ideal}\left(\operatorname{im}\left(\nabla_{A}\right)\right)$.

If $G$ is a group with $\operatorname{dim}_{\mathbb{L}} H_{1}(G ; \mathbb{k})<\infty$, we set $\mathfrak{h}(G, \mathbb{k})=\mathfrak{h}\left(H^{*}(G ; \mathbb{k})\right)$. In fact, if $X$ is any path-connected space with $G=\pi_{1}(X)$, then we may define $\mathfrak{h}(X, \mathbb{k}):=\mathfrak{h}\left(H^{*}(X ; \mathbb{k})\right)$, after which one checks that $\mathfrak{h}(X, \mathbb{k}) \cong \mathfrak{h}(G, \mathbb{k})$. Moreover, if $G$ is finitely generated, then $\mathfrak{h}(G, \mathbb{Q}) \cong \mathfrak{h}(G) \otimes \mathbb{Q}$.

On a historical note, the holonomy Lie algebra of a (finitely generated) group $G$ was first defined (over $\mathbb{k}=\mathbb{Q}$ ) by K.-T. Chen [8], and later studied by Kohno [33] in the case when $G$ is the fundamental group of the complement of a complex projective hypersurface. In [40], Markl and Papadima extended the definition of the holonomy Lie algebra to integral coefficients. Further in-depth studies were done by Papadima-Suciu [47] and Suciu-Wang [64, 65].
4.2. A comparison map. A notable fact about the holonomy Lie algebra is its relationship to the associated graded Lie algebra. This relationship is detailed in the next theorem.

Theorem 4.2 ([40, 47, 64]). For every group $G$ such that $G_{\text {abf }}$ is finitely generated, there exists a natural epimorphism of graded Lie algebras, $\Psi: \mathfrak{h}(G) \rightarrow \operatorname{gr}(G)$, which induces isomorphisms in degrees 1 and 2 and descends to epimorphisms $\Psi^{(r)}: \mathfrak{h}(G) / \mathfrak{h}(G)^{(r)} \rightarrow$ $\operatorname{gr}\left(G / G^{(r)}\right)$ for all $r \geqslant 2$.

When the group $G$ is 1-formal, the maps $\Psi \otimes \mathbb{Q}$ and $\Psi^{(r)} \otimes \mathbb{Q}$ are all isomorphisms. In general, though, the map $\Psi \otimes \mathbb{Q}$ fails to be injective, even in degree 3. Nevertheless, as we shall see next, the map $\Psi_{3}: \mathfrak{h}_{3}(G) \rightarrow \operatorname{gr}_{3}(G)$ is an isomorphism for a large class of (not necessarily 1 -formal) groups. We summarize Theorems 3.1, 4.1, and 4.3 from [50], as follows.

Theorem 4.3 ([50]). Suppose $G_{\mathrm{ab}}$ is finitely generated and torsion-free. Then,
(1) There is a natural exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathfrak{h}_{3}(G) \longrightarrow H_{2}\left(G / \gamma_{3}(G) ; \mathbb{Z}\right) \longrightarrow H_{2}(G ; \mathbb{Z}) /\left(\operatorname{ker} \nabla_{G}\right) \longrightarrow 0, \tag{4.3}
\end{equation*}
$$ where $\nabla_{G}: H_{2}(G ; \mathbb{Z}) \rightarrow H_{1}(G ; \mathbb{Z}) \wedge H_{1}(G ; \mathbb{Z})$ is the comultiplication map.

(2) If, moreover, the map $\nabla_{G}$ is injective, then, for each $k \geqslant 3$, there is a natural, split exact sequence

$$
\begin{equation*}
0 \longrightarrow \operatorname{gr}_{k}(G) \longrightarrow H_{2}\left(G / \gamma_{k}(G) ; \mathbb{Z}\right) \longrightarrow H_{2}(G ; \mathbb{Z}) /\left(\operatorname{ker} \nabla_{G}\right) \longrightarrow 0 \tag{4.4}
\end{equation*}
$$

Furthermore, the canonical projection $\Psi_{3}: \mathfrak{h}_{3}(G) \rightarrow \mathrm{gr}_{3}(G)$ is an isomorphism.
Consequently, when $G_{\text {ab }}$ is finitely generated and torsion-free and $\nabla_{G}$ is injective, the Schur multiplier of the second nilpotent quotient of $G$ decomposes as

$$
\begin{equation*}
H_{2}\left(G / \gamma_{3}(G) ; \mathbb{Z}\right) \cong H_{2}(G ; \mathbb{Z}) \oplus \mathfrak{h}_{3}(G) \tag{4.5}
\end{equation*}
$$

4.3. Infinitesimal Alexander invariant. Let $G$ be a group and assume that its torsion-free abelianization, $G_{\mathrm{abf}}=G_{\mathrm{ab}} /$ Tors, is finitely generated. Then $G_{\text {abf }}$ is a free abelian group of rank $r=b_{1}(G)$. The symmetric algebra on this group, $S=\operatorname{Sym}\left(G_{\text {abf }}\right)$, is naturally isomorphic to $\operatorname{gr}\left(\mathbb{Z}\left[G_{\text {abf }}\right]\right)$. In concrete terms, if we identify $G_{\text {abf }}$ with $\mathbb{Z}^{r}$, then $\operatorname{Sym}\left(G_{\text {abf }}\right)$ gets identified with the polynomial ring $\mathbb{Z}\left[x_{1}, \ldots, x_{r}\right]$.

A homomorphism $\alpha: G \rightarrow H$ between two groups as above induces a homomorphism $\alpha_{\text {abf }}: G_{\text {abf }} \rightarrow H_{\text {abf }}$, which extends to a ring map, $\tilde{\alpha}_{\text {abf }}: \operatorname{Sym}\left(G_{\text {abf }}\right) \rightarrow \operatorname{Sym}\left(H_{\text {abf }}\right)$. If we identify these symmetric algebras with the corresponding polynomial rings, the map $\tilde{\alpha}_{\text {abf }}$ is the linear change of variables defined by the matrix of $\alpha_{\text {abf }}$. Consequently, if $\alpha_{\text {abf }}$ is injective (respectively, surjective), then $\tilde{\alpha}_{\text {abf }}$ is also injective (respectively, surjective).

Following the approach from [47], we define the infinitesimal Alexander invariant of $G$ to be the quotient (abelian) group

$$
\begin{equation*}
\mathfrak{B}(G):=\mathfrak{h}(G)^{\prime} / \mathfrak{h}(G)^{\prime \prime}, \tag{4.6}
\end{equation*}
$$

viewed as a graded module over the ring $S=\operatorname{Sym}\left(G_{\text {abf }}\right)$. Setting $\mathfrak{h}=\mathfrak{h}(G)$, the module structure on $\mathfrak{B}(G)$ comes from the exact sequence $0 \rightarrow \mathfrak{h}^{\prime} / \mathfrak{h}^{\prime \prime} \rightarrow \mathfrak{h} / \mathfrak{h}^{\prime \prime} \rightarrow \mathfrak{h} / \mathfrak{h}^{\prime} \rightarrow 0$ via the adjoint action of $\mathfrak{h} / \mathfrak{h}^{\prime}=\mathfrak{h}_{1}=G_{\text {abf }}$ on $\mathfrak{h}^{\prime} / \mathfrak{h}^{\prime \prime}$ given by $g \cdot \bar{x}=\overline{[g, x]}$ for $g \in \mathfrak{h}_{1}$ and $x \in \mathfrak{h}^{\prime}=\oplus_{k \geqslant 2} \mathfrak{h}_{k}$. Since the grading for $S$ starts with $S_{0}=\mathbb{Z}$, we are led to define the grading on $\mathfrak{B}(G)$ as

$$
\begin{equation*}
\mathfrak{B}_{k}(G)=\left(\mathfrak{h}^{\prime} / \mathfrak{h}^{\prime \prime}\right)_{k+2}, \tag{4.7}
\end{equation*}
$$

for $k \geqslant 0$. This equality may be viewed as an infinitesimal analogue of Massey's isomorphism from Corollary 3.4.

When $G$ admits a finite, commutator-relators presentation (as happens for instance with an arrangement group), the $S$-module $\mathfrak{B}(G)$ is isomorphic to the "linearization" of the Alexander invariant $B(G)$, see [47, Prop. 9.3].

The above construction is functorial. More precisely, if $\alpha: G \rightarrow H$ is a homomorphism between two groups as above, then $\alpha$ induces a morphism of graded Lie algebras, $\mathfrak{h}(\alpha): \mathfrak{h}(G) \rightarrow \mathfrak{h}(H)$, which preserves the respective derived series. Hence, the restriction $\mathfrak{h}^{\prime}(\alpha): \mathfrak{h}(G)^{\prime} \rightarrow \mathfrak{h}(H)^{\prime}$ further restrics to a map $\mathfrak{h}^{\prime \prime}(\alpha): \mathfrak{h}(G)^{\prime \prime} \rightarrow \mathfrak{h}(H)^{\prime \prime}$, and thus induces a homomorphism $\mathfrak{B}(\alpha): \mathfrak{B}(G) \rightarrow \mathfrak{B}(H)$. A routine check shows that $\mathfrak{B}(\alpha)$ is a morphism of graded modules covering the ring map $\tilde{\alpha}_{\text {abf }}: \operatorname{Sym}\left(G_{\text {abf }}\right) \rightarrow \operatorname{Sym}\left(H_{\text {abf }}\right)$, and that $\mathfrak{B}(\beta \circ \alpha)=\mathfrak{B}(\beta) \circ \mathfrak{B}(\alpha)$. Clearly, if $\mathfrak{h}^{\prime}(\alpha): \mathfrak{h}(G)^{\prime} \rightarrow \mathfrak{h}(H)^{\prime}$ is surjective, then $\mathfrak{B}(\alpha): \mathfrak{B}(G) \rightarrow \mathfrak{B}(H)$ is also surjective, and if $\mathfrak{h}^{\prime}(\alpha)$ is an isomorphism, then $\mathfrak{B}(\alpha)$ is also
an isomorphism. Consequently, if $\alpha$ is surjective, then $\mathfrak{h}^{\prime}(\alpha)$ is surjective, and so $\mathfrak{B}(\alpha)$ is also surjective.

Denoting by $\mathfrak{B}(H)_{\alpha}$ the module obtained from $\mathfrak{B}(H)$ by restriction of scalars along $\tilde{\alpha}_{\text {abf }}$, we may view the map $\mathfrak{B}(\alpha)$ as the composite $\mathfrak{B}(G) \rightarrow \mathfrak{B}(H)_{\alpha} \rightarrow \mathfrak{B}(H)$, where the first arrow is a $\operatorname{Sym}\left(G_{\text {abf }}\right)$-linear map and the second arrow is the identity map of $\mathfrak{B}(H)$, thought of as covering the ring map $\tilde{\alpha}_{\text {abf }}$.
4.4. Chen ranks and 1-formality. Let $G$ be a group with $b_{1}(G)<\infty$. The holonomy Chen ranks of $G$ are defined as $\bar{\theta}_{k}(G)=\operatorname{dim}_{\mathbb{Q}}\left(\mathfrak{h}(G ; \mathbb{Q}) / \mathfrak{h}(G ; \mathbb{Q})^{\prime \prime}\right)_{k}$. By Theorem 4.2, we have that $\theta_{k}(G) \leqslant \bar{\theta}_{k}(G)$ for all $k \geqslant 1$, with equality for $k=1$ and 2 . Using the grading convention from (4.7) and the proof of [63, Prop. 8.1], we arrive at the following infinitesimal version of formula (3.6).

Proposition 4.4 ([63]). Let $G$ be a group with $b_{1}(G)<\infty$. Then $\bar{\theta}_{k}(G)=\operatorname{dim}_{\mathbb{Q}} \mathfrak{B}_{k-2}(G ; \mathbb{Q})$ for all $k \geqslant 2$.

As we shall see below, when the group $G$ is 1-formal, the Chen ranks and the holonomy Chen ranks of $G$ are equal. The key result towards establishing this fact (proved in [24, Thm. 5.6]) uses the 1 -formality hypothesis to construct a functorial isomorphism between the Alexander invariant and its infinitesimal version, at the level of completions (over $\mathbb{Q}$ ).

Theorem 4.5 ([24]). Let $G$ be a 1-formal group. There is then a filtration-preserving isomorphism of completed modules, $\Phi_{G}: \widehat{B(G)} \otimes \mathbb{Q} \xrightarrow{\simeq} \widehat{\mathfrak{B ( G )}} \otimes \mathbb{Q}$.

This isomorphism is functorial, in the following sense. Let $\alpha: G \rightarrow H$ be a homomorphism between two 1-formal groups, and let $B(\alpha): B(G) \rightarrow B(H)$ and $\mathfrak{B}(\alpha): \mathfrak{B}(G) \rightarrow$ $\mathfrak{B}(H)$ be the induced morphisms between the two types of Alexander invariants. The following diagram then commutes.


Passing to associated graded modules, we obtain as an immediate corollary the following result (see also [63, §8] for a different approach.)

Corollary 4.6. If $G$ is 1 -formal, then $\operatorname{gr}(B(G) \otimes \mathbb{Q}) \cong \mathfrak{B}(G) \otimes \mathbb{Q}$, as graded modules over $\operatorname{gr}\left(\mathbb{Q}\left[G_{\mathrm{ab}}\right]\right) \cong \operatorname{Sym}\left(H_{1}(G ; \mathbb{Q})\right)$.
Corollary 4.7 ([47]). If $G$ is 1 -formal, then $\theta_{k}(G)=\bar{\theta}_{k}(G)$, for all $k \geqslant 2$, and so

$$
\sum_{k \geqslant 0} \theta_{k+2}(G) t^{k}=\operatorname{Hilb}(\mathfrak{B}(G) \otimes \mathbb{Q}, t) .
$$

## 5. Complements of hyperplane arrangements

We now turn to the study of complex hyperplane arrangements. This section contains a brief review of the theory of arrangements, with an emphasis on some of the topological invariants associated to their complements.
5.1. Hyperplane arrangements. An arrangement of hyperplanes is a finite set $\mathscr{A}$ of codimension-1 linear subspaces in a finite-dimensional, complex vector space $\mathbb{C}^{d+1}$. The combinatorics of the arrangement is encoded in its intersection lattice, $L(\mathscr{A})$, that is, the poset of all intersections of hyperplanes in $\mathscr{A}$ (also known as flats), ordered by reverse inclusion, and ranked by codimension. For a flat $X=\bigcap_{H \in \mathcal{B}} H$ defined by a sub-arrangement $\mathfrak{B} \subseteq \mathscr{A}$, we let $\operatorname{rank} X=\operatorname{codim} X$; we also write $L_{k}(\mathscr{A})=\{X \in L(\mathscr{A}) \mid \operatorname{rank} X=k\}$.

Unless otherwise stated, we will assume that the arrangement is central, that is, all the hyperplanes pass through the origin. For each hyperplane $H \in \mathscr{A}$, let $f_{H}: \mathbb{C}^{d+1} \rightarrow \mathbb{C}$ be a linear form with kernel $H$. The product $f=\prod_{H \in \mathscr{A}} f_{H}$, then, is a defining polynomial for the arrangement, unique up to a non-zero constant factor. Notice that $f$ is a homogeneous polynomial of degree equal to $n=|\mathscr{A}|$, the number of hyperplanes comprising $\mathscr{A}$.

The complement of the arrangement, $M(\mathscr{A})=\mathbb{C}^{d+1} \backslash \bigcup_{H \in \mathscr{A}} H$, is a connected, smooth complex quasi-projective variety. Moreover, $M=M(\mathscr{A})$ is a Stein manifold, and thus it has the homotopy type of a CW-complex of dimension at most $d+1$. In fact, $M$ splits off the linear subspace $\bigcap_{H \in \mathscr{A}} H$; if the dimension of this subspace (which we call the corank of $\mathscr{A}$ ) is equal to 0 , we say that $\mathscr{A}$ is essential.

The group $\mathbb{C}^{*}$ acts freely on $\mathbb{C}^{d+1} \backslash\{0\}$ via $\zeta \cdot\left(z_{0}, \ldots, z_{d}\right)=\left(\zeta z_{0}, \ldots, \zeta z_{d}\right)$. The orbit space is the complex projective space of dimension $d$, while the orbit map, $\pi: \mathbb{C}^{d+1} \backslash\{0\} \rightarrow \mathbb{C P}^{d}$, $z \mapsto[z]$, is the Hopf fibration. The set $\mathbb{P}(\mathscr{A})=\{\pi(H): H \in \mathscr{A}\}$ is an arrangement of codimension 1 projective subspaces in $\mathbb{C P}^{d}$. Its complement, $U=U(\mathscr{A})$, coincides with the quotient $\mathbb{P}(M)=M / \mathbb{C}^{*}$. The Hopf map restricts to a bundle map, $\pi: M \rightarrow U$, with fiber $\mathbb{C}^{*}$. Fixing a hyperplane $H \in \mathscr{A}$, we see that $\pi$ is also the restriction to $M$ of the bundle map $\mathbb{C}^{d+1} \backslash H \rightarrow \mathbb{C P}^{d} \backslash \pi(H) \cong \mathbb{C}^{d}$. This latter bundle is trivial, and so we have a diffeomorphism $M \cong U \times \mathbb{C}^{*}$.
5.2. Cohomology ring. The cohomology ring of a hyperplane arrangement complement $M=M(\mathscr{A})$ was computed by Brieskorn in [6], building on the work of Arnol'd on the cohomology ring of the pure braid group. In [45], Orlik and Solomon gave a simple description of this ring, solely in terms of the intersection lattice $L(\mathscr{A})$, as follows. Fix a linear order on $\mathscr{A}$, and let $E=E(\mathscr{A})$ be the exterior algebra over $\mathbb{Z}$ with generators $\left\{e_{H} \mid H \in \mathscr{A}\right\}$ in degree 1. Next, define a differential $\partial: E \rightarrow E$ of degree -1 , starting from $\partial(1)=0$ and $\partial\left(e_{H}\right)=1$, and extending $\partial$ to a linear operator on $E$, using the graded Leibniz rule. Finally, let $I(\mathscr{A})$ be the ideal of $E$ generated by $\partial e_{\mathfrak{B}}$, for all sub-arrangements
$\mathfrak{B} \subset \mathscr{A}$ such that codim $\bigcap_{H \in \mathfrak{B}} H<|\mathfrak{B}|$, where $e_{\mathfrak{B}}:=\prod_{H \in \mathfrak{B}} e_{H}$. Then

$$
\begin{equation*}
H^{*}(M(\mathscr{A}) ; \mathbb{Z})=E(\mathscr{A}) / I(\mathscr{A}) \tag{5.1}
\end{equation*}
$$

Every arrangement complement $M=M(\mathscr{A})$ is formal, that is, its rational cohomology algebra can be connected by a zig-zag of quasi-isomorphisms to the algebra of polynomial differential forms on $M$ defined by Sullivan in [66]. Indeed, for each $H \in \mathscr{A}$, the 1 -form $\omega_{H}=\frac{1}{2 \pi \mathrm{i}} d \log f_{H}$ on $\mathbb{C}^{d+1}$ restricts to a 1 -form on $M$. As shown by Brieskorn [6], if $\mathscr{D}$ denotes the subalgebra of the de Rham algebra $\Omega_{\mathrm{dR}}^{*}(M)$ generated over $\mathbb{R}$ by these 1 forms, the correspondence $\omega_{H} \mapsto\left[\omega_{H}\right]$ induces an isomorphism $\mathscr{D} \rightarrow H^{*}(M ; \mathbb{R})$, and then Sullivan's machinery implies that $M$ is formal.
5.3. Localized sub-arrangements. The localization of an arrangement $\mathscr{A}$ at a flat $X \in$ $L(\mathscr{A})$ is the sub-arrangement $\mathscr{A}_{X}:=\{H \in \mathscr{A}: H \supset X\}$. The inclusion $\mathscr{A}_{X} \subset \mathscr{A}$ gives rise to an inclusion of complements, $j_{X}: M(\mathscr{A}) \hookrightarrow M\left(\mathscr{A}_{X}\right)$. The inclusions $\left\{j_{X}\right\}_{X \in L(\mathscr{A})}$ assemble into a map

$$
\begin{equation*}
j: M(\mathscr{A}) \longrightarrow \prod_{X \in L(\mathscr{A})} M\left(\mathscr{A}_{X}\right) \tag{5.2}
\end{equation*}
$$

As shown by Brieskorn [6, Lemma 3] (see also [46, Lemma 5.91]) the homomorphism induced in cohomology by $j$ is an isomorphism in each degree $k \geqslant 0$. Moreover, the groups $H^{k}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$ are torsion-free, and so, by the Künneth formula, we have isomorphisms

$$
\begin{equation*}
H^{k}(M(\mathscr{A}) ; \mathbb{Z}) \cong \bigoplus_{X \in L_{k}(\mathscr{A})} H^{k}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right) \tag{5.3}
\end{equation*}
$$

for all $k \geqslant 0$. Likewise, the Orlik-Solomon ideal decomposes in each degree as $I^{k}(\mathscr{A}) \cong$ $\oplus_{X \in L_{k}(\mathscr{A})} I^{k}\left(\mathscr{A}_{X}\right)$. It follows that the homology groups of the complement of $\mathscr{A}$ are torsionfree, with ranks given by

$$
\begin{equation*}
b_{k}(M(\mathscr{A}))=\sum_{X \in L_{k}(\mathscr{A})}(-1)^{k} \mu(X), \tag{5.4}
\end{equation*}
$$

where $\mu: L(\mathscr{A}) \rightarrow \mathbb{Z}$ is the Möbius function of the intersection lattice, defined inductively by $\mu\left(\mathbb{C}^{n}\right)=1$ and $\mu(X)=-\sum_{Z \ni X} \mu(Z)$. In particular, $H_{1}(M(\mathscr{A}) ; \mathbb{Z})$ is free abelian of rank equal to the cardinality of the arrangement, $|\mathscr{A}|$.

As noted in [51] and [42], the fact that $H^{*}(M(\mathscr{A}) ; \mathbb{Z})$ is torsion-free (as a group) and generated in degree 1 (as an algebra) implies that, for each $2 \leqslant i \leqslant d+1$, the Hurewicz homomorphism $h: \pi_{i}(M(\mathscr{A})) \rightarrow H_{i}(M(\mathscr{A}) ; \mathbb{Z})$ is the zero map.
5.4. Fundamental group. Fix a basepoint $x_{0}$ in the complement of $\mathscr{A}$, and consider the fundamental group $G(\mathscr{A})=\pi_{1}\left(M(\mathscr{A}), x_{0}\right)$. For each hyperplane $H \in \mathscr{A}$, pick a meridian curve about $H$, oriented compatibly with the complex orientations on $\mathbb{C}^{d+1}$ and $H$, and let $x_{H}$ denote the based homotopy class of this curve, joined to the basepoint by a path in $M$. By the van Kampen theorem, then, the arrangement group, $G=G(\mathscr{A})$, is generated by the
set $\left\{x_{H}: H \in \mathscr{A}\right\}$. Using the braid monodromy algorithm from [16], one may obtain a finite presentation of the form $G=F_{n} / R$, where $F_{n}$ is the rank $n$ free group on the set of meridians and the relators in $R$ belong to the commutator subgroup $F_{n}^{\prime}$, so that $G_{\mathrm{ab}}=H_{1}(G ; \mathbb{Z}) \cong \mathbb{Z}^{n}$.

Probably the best-known example is the braid arrangement $\mathscr{A}_{n}$, consisting of the diagonal hyperplanes in $\mathbb{C}^{n}$. It is readily seen that $L\left(\mathscr{A}_{n}\right)$ is the lattice of partitions of $[n]=\{1, \ldots, n\}$, ordered by refinement, while $M\left(\mathscr{A}_{n}\right)$ is the configuration space $F(\mathbb{C}, n)$ of $n$ ordered points in $\mathbb{C}$, which is a classifying space for $P_{n}$, the pure braid group on $n$ strings.

More generally, if the intersection lattice $L(\mathscr{A})$ is supersolvable, the complement $M=$ $M(\mathscr{A})$ is a classifying space for the group $G=G(\mathscr{A})$. On the other hand, there are many arrangements $\mathscr{A}$ (for instance, generic arrangements) for which $M$ is not a $K(G, 1)$. Nevertheless, as noted above, the Hurewicz map $h: \pi_{2}(M) \rightarrow H_{2}(M ; \mathbb{Z})$ is the zero map. Therefore, if $g: M \rightarrow K(G, 1)$ is a classifying map, then, by the Hopf exact sequence, the induced homomorphism $g_{*}: H_{i}(M ; \mathbb{Z}) \rightarrow H_{i}(G ; \mathbb{Z})$ is an isomorphism for $i \leqslant 2$.

For the purpose of computing the group $G=G(\mathscr{A})$, it is enough to assume that the arrangement $\mathscr{A}$ lives in $\mathbb{C}^{3}$, in which case $\overline{\mathscr{A}}=\mathbb{P}(\mathscr{A})$ is an arrangement of (projective) lines in $\mathbb{C P}^{2}$. This is clear when the rank of $\mathscr{A}$ is at most 2 , and may be achieved otherwise by taking a generic 3 -slice, an operation which does not change either the poset $L_{\leqslant 2}(\mathscr{A})$ or the group $G$. For a rank-3 arrangement, the set $L_{1}(\mathscr{A})$ is in 1-to-1 correspondence with the lines of $\mathscr{A}$, while $L_{2}(\mathscr{A})$ is in 1-to-1 correspondence with the intersection points of $\mathscr{A}$. Moreover, the poset structure of $L_{\leqslant 2}(\mathscr{A})$ mirrors the incidence structure of the point-line configuration $\overline{\mathscr{A}}$.
5.5. Localization and fundamental groups. Choosing a point $x_{0}$ sufficiently close to $\mathbf{0} \in \mathbb{C}^{d+1}$, we can make $x_{0}$ a common basepoint for both $M(\mathscr{A})$ and all the local complements $M\left(\mathscr{A}_{X}\right)$. The next result (Lemmas 4.1 and 4.3 from [21]) gives a rough analog of Brieskorn's Lemma at the level of fundamental groups. For each flat $X \in L(\mathscr{A})$, let $j_{X}: M(\mathscr{A}) \rightarrow M\left(\mathscr{A}_{X}\right)$ be the corresponding inclusion.

Lemma 5.1 ([21]). There exist basepoint-preserving maps $r_{X}: M\left(\mathscr{A}_{X}\right) \rightarrow M(\mathscr{A})$ such that $j_{X} \circ r_{X} \simeq \mathrm{id}$ relative to $x_{0}$. Moreover, if $H \in \mathscr{A}$ and $H \not \supset X$, then the composite $r_{X} \circ j_{X} \circ r_{H}$ is null-homotopic.

Setting $G\left(\mathscr{A}_{X}\right)=\pi_{1}\left(M\left(\mathscr{A}_{X}\right), x_{0}\right)$, it follows from the lemma that the induced homomorphism $\left(j_{X}\right)_{\sharp}: G(\mathscr{A}) \rightarrow G\left(\mathscr{A}_{X}\right)$ is surjective and $\left(r_{X}\right)_{\sharp}: G\left(\mathscr{A}_{X}\right) \rightarrow G(\mathscr{A})$ is injective.

Of particular interest to us is what happens when $X$ has codimension 2. For a 2-flat $X$, the localized sub-arrangement $\mathscr{A}_{X}$ is a pencil of $|X|=\mu(X)+1$ hyperplanes. Consequently, $M\left(\mathscr{A}_{X}\right)$ is homeomorphic to $(\mathbb{C} \backslash\{\mu(X)$ points $\}) \times \mathbb{C}^{*} \times \mathbb{C}^{d-1}$, and so $M\left(\mathscr{A}_{X}\right)$ is a classifying space for the group

$$
\begin{equation*}
G\left(\mathscr{A}_{X}\right) \cong F_{\mu(X)} \times \mathbb{Z} \tag{5.5}
\end{equation*}
$$

5.6. The second nilpotent quotient. Let $G=G(\mathscr{A})$ be an arrangement group. Plainly, the abelianization $G_{\mathrm{ab}}=H_{1}(M(\mathscr{A}) ; \mathbb{Z})$ is the free abelian group on $\left\{x_{H}\right\}_{H \in \mathscr{A}}$. On the other hand, as noted for instance in [42], the abelian group $\operatorname{gr}_{2}(G)$ is the $\mathbb{Z}$-dual of $I^{2}(\mathscr{A})$, the degree-2 part of the Orlik-Solomon ideal. In particular, $\mathrm{gr}_{2}(G)$ is also torsion-free.

Proposition 5.2 ([42]). For any arrangement $\mathscr{A}$, the second nilpotent quotient of $G(\mathscr{A})$ fits into a central extension of the form

$$
\begin{equation*}
0 \longrightarrow\left(I^{2}(\mathscr{A})\right)^{\vee} \longrightarrow G(\mathscr{A}) / \gamma_{3}(G(\mathscr{A})) \longrightarrow H_{1}(M(\mathscr{A}) ; \mathbb{Z}) \longrightarrow 0 . \tag{5.6}
\end{equation*}
$$

Furthermore, the $k$-invariant of this extension, $\chi_{2}: H_{2}\left(G_{\mathrm{ab}} ; \mathbb{Z}\right) \rightarrow \mathrm{gr}_{2}(G)$, is the dual of the inclusion map $I^{2}(\mathscr{A}) \hookrightarrow E^{2}(\mathscr{A})=\bigwedge^{2} G_{\mathrm{ab}}$.

Let $F=F(\mathscr{A})$ be the free group on generators $\left\{x_{H}: H \in \mathscr{A}\right\}$. It follows that $G / \gamma_{3}(G)$ is the quotient of the free, 2-step nilpotent group $F / \gamma_{3}(F)$ by all commutation relations of the form

$$
\begin{equation*}
r_{H, X}:=\left[x_{H}, \prod_{\substack{K \in \mathscr{A} \\ K \supset X}} x_{K}\right], \tag{5.7}
\end{equation*}
$$

indexed by pairs of hyperplanes $H \in \mathscr{A}$ and flats $X \in L_{2}(\mathscr{A})$ such that $H \supset X$ (see [52, 42]). From this description, it is apparent that the second nilpotent quotient of an arrangement group is combinatorially determined. More precisely, suppose $\mathscr{A}$ and $\mathscr{B}$ are two arrangements such that $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathscr{B})$, meaning, there are bijections $\alpha: \mathscr{A} \rightarrow \mathscr{B}$ and $\beta: L_{2}(\mathscr{A}) \rightarrow L_{2}(\mathscr{B})$ such that $\beta\left(H_{1} \cap \cdots \cap H_{r}\right)=\alpha\left(H_{1}\right) \cap \cdots \cap \alpha\left(H_{r}\right)$. Then $\alpha$ extends to an isomorphism of free groups, $\alpha: F(\mathscr{A}) \rightarrow F(\mathscr{B})$, which sends $r_{H, X} \in$ $F(\mathscr{A})$ to $r_{\alpha(H), \beta(X)} \in F(\mathscr{B})$, and thus descends to an isomorphism $G(\mathscr{A}) / \gamma_{3}(G(\mathscr{A})) \xrightarrow{\simeq}$ $G(\mathscr{B}) / \gamma_{3}(G(\mathscr{B}))$.

## 6. Holonomy Lie algebras of arrangements

We now study the holonomy Lie algebra $\mathfrak{h}(\mathscr{A})$ of an arrangement group $G(\mathscr{A})$, how it behaves under certain operations with arrangements, and how it relates to the associated graded Lie algebra $\operatorname{gr}(G(\mathscr{A}))$.
6.1. Holonomy Lie algebra. The holonomy Lie algebra of an arrangement $\mathscr{A}$ is defined as the holonomy Lie algebra of the fundamental group $G(\mathscr{A})=\pi_{1}(M(\mathscr{A}))$ of the complement of $\mathscr{A}$,

$$
\begin{equation*}
\mathfrak{h}(\mathscr{A}):=\mathfrak{h}(G(\mathscr{A})) . \tag{6.1}
\end{equation*}
$$

An explicit presentation for this Lie algebra was given by Kohno [33], as follows. Let $\operatorname{Lie}(\mathscr{A}):=\operatorname{Lie}\left(H_{1}(M(\mathscr{A}) ; \mathbb{Z})\right)$ be the free Lie algebra on variables $\left\{x_{H}\right\}_{H \in \mathscr{A}}$. From the

Orlik-Solomon description of the cohomology ring $H^{*}(M(\mathscr{A}) ; \mathbb{Z})$ as the quotient $E / I(\mathscr{A})$, it follows that

$$
\begin{equation*}
\mathfrak{h}(\mathscr{A})=\operatorname{Lie}(\mathscr{A}) / J(\mathscr{A}), \tag{6.2}
\end{equation*}
$$

where $J(\mathscr{A})$ is the Lie ideal generated by all the Lie brackets of the form

$$
\begin{equation*}
\mathfrak{r}_{H, X}:=\left[x_{H}, \sum_{\substack{K \in \mathscr{A} \\ K \supset X}} x_{K}\right], \tag{6.3}
\end{equation*}
$$

indexed by pairs of hyperplanes $H \in \mathscr{A}$ and flats $X \in L_{2}(\mathscr{A})$ such that $H \supset X$. Plainly, $J(\mathscr{A})$ is a homogeneous ideal, and so $\mathfrak{h}(\mathscr{A})$ inherits the structure of a graded Lie algebra from $\operatorname{Lie}(\mathscr{A})$. In fact, $\mathfrak{h}(\mathscr{A})$ is a finitely generated, quadratic Lie algebra, with generators $x_{H}$ in degree 1 and relators $\mathfrak{r}_{H, X}$ in degree 2 . Completely analogous considerations apply to the holonomy $\mathbb{k}$-Lie algebra $\mathfrak{h}(\mathscr{A}, \mathbb{k}):=\mathfrak{h}(G(\mathscr{A}), \mathbb{k})$ over a field $\mathbb{k}$.

Example 6.1. Let $\mathscr{A}$ be a pencil of $n$ lines through the origin of $\mathbb{C}^{2}$. Then $\mathfrak{h}(\mathscr{A})$ is the quotient of the free $\operatorname{Lie} \operatorname{algebra} \operatorname{Lie}(n)=\operatorname{Lie}\left(x_{1}, \ldots, x_{n}\right)$ by the ideal generated by the relators $\left[x_{i}, \sum_{j=1}^{n} x_{j}\right]$ for $1 \leqslant i<n$. Clearly, $\mathfrak{h}(\mathscr{A}) \cong \operatorname{Lie}(n-1) \times \operatorname{Lie}(1)$. A basis for $\mathfrak{h}_{2}(\mathscr{A})$ consists of all the brackets $\left[x_{i}, x_{j}\right]$ with $1 \leqslant i<j<n$, while a basis for $\mathfrak{h}_{3}(\mathscr{A})$ consists of all the triple brackets $\left[x_{i},\left[x_{j}, x_{k}\right]\right]$ with $1 \leqslant i, j, k<n, i \geqslant j$, and $j<k$. $\diamond$

From the presentation of the holonomy Lie algebra of an arrangement $\mathscr{A}$ given in (6.2)(6.3), it follows that $\mathfrak{b}(\mathscr{A})$ depends only on the underlying matroid of the arrangement, and in fact, only on the ranked poset $L_{\leqslant 2}(\mathscr{A})$. Indeed, suppose $\mathscr{A}$ and $\mathscr{B}$ are two arrangements such that there exist compatible bijections $\alpha: \mathscr{A} \rightarrow \mathscr{B}$ and $\beta: L_{2}(\mathscr{A}) \rightarrow L_{2}(\mathscr{B})$. Then $\alpha$ extends to a isomorphism between the respective free $\operatorname{Lie}$ algebras, $\operatorname{Lie}(\alpha): \operatorname{Lie}(\mathscr{A}) \xrightarrow{\simeq}$ $\operatorname{Lie}(\mathscr{B})$, which sends $\mathfrak{r}_{H, X} \in \operatorname{Lie}_{2}(\mathscr{A})$ to $\mathfrak{r}_{\alpha(H), \beta(X)} \in \operatorname{Lie}_{2}(\mathscr{B})$. Therefore, $\operatorname{Lie}(\alpha)$ descends to an isomorphism $\mathfrak{h}(\alpha): \mathfrak{h}(\mathscr{A}) \xrightarrow{\simeq} \mathfrak{h}(\mathscr{B})$ of graded Lie algebras.
6.2. A comparison map. By Theorem 4.2, there is a surjective Lie algebra map, $\Psi=$ $\Psi_{\mathscr{A}}: \mathfrak{h}(\mathscr{A}) \rightarrow \operatorname{gr}(G(\mathscr{A}))$. The 1-formality of the group $G(\mathscr{A})$ implies that the associated graded Lie algebra $\operatorname{gr}(G(\mathscr{A}))$ and the holonomy Lie algebra $\mathfrak{b}(\mathscr{A})$ are rationally isomorphic; therefore, the map

$$
\begin{equation*}
\Psi \otimes \mathbb{Q}: \mathfrak{h}(\mathscr{A}) \otimes \mathbb{Q} \longrightarrow \operatorname{gr}(G(\mathscr{A})) \otimes \mathbb{Q} \tag{6.4}
\end{equation*}
$$

is an isomorphism. Consequently, all the LCS ranks of an arrangement group $G(\mathscr{A})$ are determined by the truncated intersection lattice $L_{\leqslant 2}(\mathscr{A})$, as follows:

$$
\begin{equation*}
\phi_{k}(G(\mathscr{A}))=\operatorname{dim}_{\mathbb{Q}} \mathfrak{h}_{k}(\mathscr{A}) \otimes \mathbb{Q} . \tag{6.5}
\end{equation*}
$$

In general, there exist arrangements $\mathscr{A}$ for which the map $\Psi_{\mathscr{A}}$ is not injective. Nevertheless, we have the following result from [50], that shows this cannot happen in degree $k=3$; for completeness, we provide a quick proof.

Theorem 6.2 ([50]). For an arrangement $\mathscr{A}$ with complement $M=M(\mathscr{A})$ and group $G=\pi_{1}(M)$, the following hold.
(1) $H_{2}\left(G / \gamma_{3}(G) ; \mathbb{Z}\right) \cong H_{2}(M ; \mathbb{Z}) \oplus \mathfrak{h}_{3}(\mathscr{A})$.
(2) The map $\Psi_{3}: \mathfrak{h}_{3}(\mathscr{A}) \rightarrow \operatorname{gr}_{3}(G)$ is an isomorphism.

Proof. Since $H^{*}(M ; \mathbb{Z})$ is generated in degree 1 , the cup-product map $\cup_{M}: H^{1}(M ; \mathbb{Z}) \wedge$ $H^{1}(M ; \mathbb{Z}) \rightarrow H^{2}(M ; \mathbb{Z})$ is surjective; thus the comultiplication map $\nabla_{M}: H_{2}(M ; \mathbb{Z}) \rightarrow$ $H_{1}(M ; \mathbb{Z}) \wedge H_{1}(M ; \mathbb{Z})$ is injective. Moreover, as noted previously, any classifying map $g: M \rightarrow K(G, 1)$ induces an isomorphism on $H_{\leqslant 2}$; hence, by naturality of cup-products, the map $\nabla_{G}$ is also injective. Both claims now follow from Theorem 4.3.

Consequently, the group $\operatorname{gr}_{3}(G(\mathscr{A}))$ is combinatorially determined; that is, if $\mathscr{A}$ and $\mathfrak{B}$ are two arrangements such that $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathfrak{B})$, then $\operatorname{gr}_{3}(G(\mathscr{A})) \cong \operatorname{gr}_{3}(G(\mathfrak{B}))$. These considerations naturally lead to the following question.

Question 6.3. For an arrangement $\mathscr{A}$, is the (finitely generated abelian) group $\mathfrak{h}_{3}(\mathscr{A})=$ $\operatorname{gr}_{3}(G(\mathscr{A}))$ torsion-free?

In view of Theorem 6.2 and the fact that $H_{*}(M(\mathscr{A}) ; \mathbb{Z})$ is torsion-free, the question may be rephrased as: Is the Schur multiplier of the finitely generated, 2-step nilpotent group $G(\mathscr{A}) / \gamma_{3}(G(\mathscr{A}))$ torsion-free?

Remark 6.4. In [25], Falk sketched the construction of a Sullivan 1-minimal model for the complement of an arrangement $\mathscr{A}$, and used this to show that $\phi_{3}(G(\mathscr{A}))$ is equal to the nullity of the multiplication map $E^{1} \otimes I^{2} \rightarrow E^{3}$ over $\mathbb{Q}$. Further information on the ranks of the LCS quotients of an arrangement group can be found in [56, 55].

Remark 6.5. As first noted in [56], there exist arrangements $\mathscr{A}$ for which $\operatorname{gr}_{k}(G(\mathscr{A}))$ has non-zero torsion for some $k>3$. This naturally raised the question whether such torsion in the LCS quotients of arrangement groups is combinatorially determined. The question was recently answered in the negative by Artal Bartolo, Guerville-Ballé, and ViuSos [4], who produced a pair of arrangements $\mathscr{A}$ and $\mathscr{B}$ with $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathscr{B})$, yet $\operatorname{gr}_{4}(G(\mathscr{A})) \not \equiv \operatorname{gr}_{4}(G(\mathscr{B}))$; the difference lies in the 2-torsion of the respective groups.
6.3. Products of arrangements. Let us recall a definition from [46]. If $\mathscr{A}$ is an arrangement in $\mathbb{C}^{r}$ and $\mathscr{B}$ is an arrangement in $\mathbb{C}^{s}$, then their product, $\mathscr{A} \times \mathscr{B}$, is the arrangement in $\mathbb{C}^{r+s}=\mathbb{C}^{r} \times \mathbb{C}^{s}$ given by

$$
\begin{equation*}
\mathscr{A} \times \mathscr{B}:=\left\{H \times \mathbb{C}^{s}: H \in \mathscr{A}\right\} \cup\left\{\mathbb{C}^{r} \times K: K \in \mathscr{B}\right\} . \tag{6.6}
\end{equation*}
$$

It is readily seen that the intersection lattice of the product arrangement is isomorphic to the product of the respective intersection posets, that is,

$$
\begin{equation*}
L(\mathscr{A} \times \mathscr{B}) \cong L(\mathscr{A}) \times L(\mathscr{B}) \tag{6.7}
\end{equation*}
$$

Alternatively, if $f=f\left(z_{1}, \ldots, z_{r}\right)$ and $g=g\left(w_{1}, \ldots, w_{s}\right)$ are defining polynomials for $\mathscr{A}$ and $\mathscr{B}$, respectively, then a defining polynomial for $\mathscr{A} \times \mathscr{B}$ is $k\left(z_{1}, \ldots, z_{r}, w_{1}, \ldots, w_{s}\right):=$ $f\left(z_{1}, \ldots, z_{r}\right) g\left(w_{1}, \ldots, w_{s}\right)$. Plainly, $k(z, w) \neq 0$ if and only if $f(z) \neq 0$ and $g(w) \neq 0$; thus, the complement of the product arrangement, $M(\mathscr{A} \times \mathscr{B})$, is diffeomorphic to the product of the complements of the two arrangements, $M(\mathscr{A}) \times M(\mathscr{B})$. Consequently, the group $G(\mathscr{A} \times \mathscr{B})$ is isomorphic to the direct product $G(\mathscr{A}) \times G(\mathscr{B})$, and it follows from Lemma 4.1 that

$$
\begin{equation*}
\mathfrak{h}(\mathscr{A} \times \mathscr{B}) \cong \mathfrak{h}(\mathscr{A}) \times \mathfrak{h}(\mathscr{B}) . \tag{6.8}
\end{equation*}
$$

6.4. Maps between holonomy Lie algebras. We now review some constructions from [48] putting them into the context that will be needed here. For an arrangement $\mathscr{A}$, we denote by $\mathbb{Z}^{\mathscr{A}}$ the free abelian group on $\mathscr{A}$, with basis $\left\{x_{H}\right\}_{H \in \mathscr{A}}$, and we let $\operatorname{Lie}(\mathscr{A})=$ $\operatorname{Lie}\left(\mathbb{Z}^{\mathscr{Q}}\right)$ be the free Lie algebra on this group. For each sub-arrangement $\mathscr{B} \subset \mathscr{A}$, we define two maps between the respective holonomy Lie algebras (in opposite directions), as follows.

First, let $\pi_{\mathscr{B}}: \mathbb{Z}^{\mathscr{A}} \rightarrow \mathbb{Z}^{\mathscr{B}}$ be the canonical projection map, defined by

$$
\pi_{\mathfrak{B}}\left(x_{H}\right)= \begin{cases}x_{H} & \text { if } H \in \mathscr{B}  \tag{6.9}\\ 0 & \text { otherwise }\end{cases}
$$

and let $\operatorname{Lie}\left(\pi_{\mathscr{B}}\right): \operatorname{Lie}(\mathscr{A}) \rightarrow \operatorname{Lie}(\mathscr{B})$ be its extension to free Lie algebras. Clearly, $\operatorname{Lie}\left(\pi_{\mathscr{B}}\right)$ takes an element $\mathrm{r}_{H, X}^{\mathscr{A}} \in J(\mathscr{A})$ as in (6.3) to the corresponding element $\mathrm{r}_{H, X}^{\mathscr{B}} \in J(\mathscr{B})$. Thus, this maps descends to an epimorphism of Lie algebras,

$$
\begin{equation*}
\mathfrak{h}\left(\pi_{\mathscr{B}}\right): \mathfrak{h}(\mathscr{A}) \longrightarrow \mathfrak{h}(\mathscr{B}) . \tag{6.10}
\end{equation*}
$$

Next, let $\iota_{\mathscr{B}}: \mathbb{Z}^{\mathscr{B}} \rightarrow \mathbb{Z}^{\mathscr{A}}$ be the canonical inclusion, given by $\iota_{\mathscr{B}}\left(x_{H}\right)=x_{H}$, and let $\operatorname{Lie}\left(\iota_{\mathscr{B}}\right): \operatorname{Lie}(\mathscr{B}) \rightarrow \operatorname{Lie}(\mathscr{A})$ be its extension to free Lie algebras. In general, this map does not take $J(\mathscr{B})$ to $J(\mathscr{A})$. However, suppose $\mathscr{B}$ is a closed sub-arrangement of $\mathscr{A}$, that is, the only linear combinations of defining forms for the hyperplanes in $\mathscr{B}$ which are defining forms for hyperplanes in $\mathscr{A}$ are (up to constants) the defining forms for the hyperplanes in $\mathscr{B}$. Then $L_{2}(\mathscr{B})=\left\{X \in L_{2}(\mathscr{A}) \mid \mathscr{A}_{X} \subset \mathscr{B}\right\}$. Thus, $\operatorname{Lie}\left(\iota_{\mathscr{B}}\right)(J(\mathscr{B})) \subset J(\mathscr{A})$, and so we obtain a map of graded Lie algebras,

$$
\begin{equation*}
\mathfrak{h}\left(\iota_{\mathscr{B}}\right): \mathfrak{h}(\mathscr{A}) \longrightarrow \mathfrak{h}(\mathscr{B}) . \tag{6.11}
\end{equation*}
$$

6.5. Local holonomy Lie algebras. Now let $X$ be a rank-2 flat in $L_{2}(\mathscr{A})$. Clearly, the localized arrangement $\mathscr{A}_{X}$ is a closed sub-arrangement of $\mathscr{A}$. Setting $\pi_{X}=\pi_{\mathscr{A}_{X}}$ and $\iota_{X}=\iota_{\mathscr{A} X}$, we obtain a pair of maps between the respective holonomy Lie algebras, $\mathfrak{h}\left(\pi_{X}\right): \mathfrak{h}(\mathscr{A}) \rightarrow$ $\mathfrak{h}\left(\mathscr{A}_{X}\right)$ and $\mathfrak{h}\left(\iota_{X}\right): \mathfrak{h}\left(\mathscr{A}_{X}\right) \rightarrow \mathfrak{h}(\mathscr{A})$. Plainly, $\pi_{X} \circ \iota_{X}$ is the identity of $\mathbb{Z}^{\mathscr{A}_{X}}$, and so $\mathfrak{h}\left(\pi_{X}\right) \circ \mathfrak{h}\left(\iota_{X}\right)$ is the identity of $\mathfrak{h}\left(\mathscr{A}_{X}\right)$. Furthermore, it is shown in [48, Lemma 3.1] that $\mathfrak{h}^{\prime}\left(\pi_{X}\right) \circ \mathfrak{h}^{\prime}\left(\iota_{Y}\right)=0$ if $Y$ is a 2-flat different from $X$.

Recall that we also have pointed maps $j_{X}: M(\mathscr{A}) \hookrightarrow M\left(\mathscr{A}_{X}\right)$ and $r_{X}: M\left(\mathscr{A}_{X}\right) \rightarrow M(\mathscr{A})$. Let $\left(r_{X}\right)_{\sharp}: G\left(\mathscr{A}_{X}\right) \rightarrow G(\mathscr{A})$ and $\left(j_{X}\right)_{\sharp}: G(\mathscr{A}) \rightarrow G\left(\mathscr{A}_{X}\right)$ be the induced a homomorphisms on fundamental groups, and let $\mathfrak{h}\left(\left(r_{X}\right)_{\sharp}\right): \mathfrak{h}\left(\mathscr{A}_{X}\right) \rightarrow \mathfrak{h}(\mathscr{A})$ and $\mathfrak{h}\left(\left(j_{X}\right)_{\sharp}\right): \mathfrak{h}(\mathscr{A}) \rightarrow \mathfrak{h}\left(\mathscr{A}_{X}\right)$ be the corresponding maps on holonomy Lie algebras.

The connection between the various maps defined above is made by the following lemma.
Lemma 6.6. For every $X \in L_{2}(\mathscr{A})$, the following hold.
(1) The homomorphism $\left(j_{X}\right)_{*}: H_{1}(M(\mathscr{A}) ; \mathbb{Z}) \rightarrow H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$ may be identified with the homomorphism $\pi_{X}: \mathbb{Z}^{\mathscr{A}} \rightarrow \mathbb{Z}^{\mathscr{A}_{X}}$.
(2) The homomorphism $\left(r_{X}\right)_{*}: H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right) \rightarrow H_{1}(M(\mathscr{A}) ; \mathbb{Z})$ may be identified with the homomorphism $\iota_{X}: \mathbb{Z}^{\mathscr{S}_{X}} \rightarrow \mathbb{Z}^{\mathscr{A}}$.
(3) $\mathfrak{h}\left(\left(j_{X}\right)_{\sharp}\right)=\mathfrak{h}\left(\pi_{X}\right)$ and $\mathfrak{h}\left(\left(r_{X}\right)_{\sharp}\right)=\mathfrak{h}\left(\iota_{X}\right)$.

Proof. The homomorphism $\left(j_{X}\right)_{\sharp}$ sends a generator $x_{H} \in G(\mathscr{A})$ to the corresponding generator $x_{H, X} \in G\left(\mathscr{A}_{X}\right)$ if $H \supset X$ and sends it to 0 if $H \neq X$, while the homomorphism $\left(r_{X}\right)_{\#}$ sends $x_{H, X} \in G\left(\mathscr{A}_{X}\right)$ to $x_{H} \in G(\mathscr{A})$. The first two claims now follow from the way the maps $\pi_{X}$ and $\iota_{X}$ were defined. Claim (3) follows from (1) and (2).

As we saw in (5.5), for each 2-flat $X \in L_{2}(\mathscr{A})$, the group $G\left(\mathscr{A}_{X}\right)$ is isomorphic to $F_{\mu(X)} \times \mathbb{Z}$; therefore, $\operatorname{gr}\left(G\left(\mathscr{A}_{X}\right)\right) \cong \operatorname{Lie}(\mu(X)) \times \operatorname{Lie}(1)$. Furthermore, as noted in Example 6.1, the Lie algebra $\mathfrak{h}\left(\mathscr{A}_{X}\right)$ is also isomorphic to $\operatorname{Lie}(\mu(X)) \times \operatorname{Lie}(1)$. Consequently, the map $\Psi_{\mathscr{A}_{X}}: \mathfrak{h}\left(\mathscr{A}_{X}\right) \rightarrow \operatorname{gr}\left(G\left(\mathscr{A}_{X}\right)\right)$ is an isomorphism.

## 7. Decomposable arrangements

In this section, we focus on a class of arrangements $\mathscr{A}$ for which several of the algebraic invariants of the group $G(\mathscr{A})$ discussed previously "decompose" in terms of the corresponding invariants of the groups $G\left(\mathscr{A}_{X}\right)$, indexed by the 2-flats in $L_{2}(\mathscr{A})$.
7.1. Local to global maps. Let $j: M(\mathscr{A}) \rightarrow \prod_{X \in L(\mathscr{A})} M\left(\mathscr{A}_{X}\right)$ be the map from (5.2). Projecting onto the factors corresponding to rank 2 flats we obtain a map

$$
\begin{equation*}
j: M(\mathscr{A}) \longrightarrow \prod_{X \in L_{2}(\mathscr{A})} M\left(\mathscr{A}_{X}\right) \tag{7.1}
\end{equation*}
$$

so that projection onto an $X$-factor composed with $j$ coincides with the map $j_{X}: M(\mathscr{A}) \hookrightarrow$ $M\left(\mathscr{A}_{X}\right)$ induced by the inclusion $\mathscr{A}_{X} \subset \mathscr{A}$. In what follows, we will identify the group $H_{1}\left(\prod_{X} M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$ with $\bigoplus_{X} H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$.
Lemma 7.1. The homomorphism $j_{*}: H_{1}(M(\mathscr{A}) ; \mathbb{Z}) \rightarrow \bigoplus_{X \in L_{2}(\mathscr{A})} H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$ is injective.

Proof. The group $H_{1}(M(\mathscr{A}) ; \mathbb{Z})$ is free abelian on generators $\left\{x_{H}: H \in \mathscr{A}\right\}$, whereas $H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$ is free abelian on generators $\left\{x_{H, X}: H \in \mathscr{A}, H \supset X\right\}$. In these bases, we have that $j_{*}\left(x_{H}\right)=\sum_{X: X \subset H} x_{H, X}$. Since every hyperplane $H \in \mathscr{A}$ contains a flat $X \in L_{2}(\mathscr{A})$, the matrix of $j_{*}$ has a minor of size $n=|\mathscr{A}|$ equal to 1 , and so the matrix has maximal rank. Hence, $j_{*}$ is injective.

The inclusions $j_{X}: M(\mathscr{A}) \hookrightarrow M\left(\mathscr{A}_{X}\right)$ induce homomorphisms $\left(j_{X}\right)_{\sharp}: G(\mathscr{A}) \rightarrow G\left(\mathscr{A}_{X}\right)$ on fundamental groups, which in turn induce maps $\mathfrak{h}\left(\left(j_{X}\right)_{\sharp}\right): \mathfrak{h}(\mathscr{A}) \rightarrow \mathfrak{h}\left(\mathscr{A}_{X}\right)$ between the corresponding holonomy Lie algebras. Let us define a "local" version of the holonomy Lie algebra by setting

$$
\begin{equation*}
\mathfrak{h}(A)^{\mathrm{loc}}:=\mathfrak{h}\left(\prod_{X \in L_{2}(\mathscr{A})} G\left(\mathscr{A}_{X}\right)\right)=\prod_{X \in L_{2}(\mathscr{A})} \mathfrak{h}\left(\mathscr{A}_{X}\right) . \tag{7.2}
\end{equation*}
$$

The maps $\mathfrak{h}\left(\left(j_{X}\right)_{\sharp}\right)$ then assemble into a map to $\mathfrak{h}\left(j_{\sharp}\right): \mathfrak{h}(\mathscr{A}) \rightarrow \mathfrak{h}(A)^{\text {loc }}$. By Lemma 6.6, this map coincides with the map $\pi=\left(\mathfrak{h}\left(\pi_{X}\right)\right)_{X}: \mathfrak{h}(\mathscr{A}) \rightarrow \prod_{X} \mathfrak{h}\left(\mathscr{A}_{X}\right)$ from [48, (2.2)].

Likewise, the maps $\mathfrak{h}\left(\iota_{X}\right)=\mathfrak{h}\left(\left(r_{X}\right)_{\sharp}\right): \mathfrak{h}\left(\mathscr{A}_{X}\right) \rightarrow \mathfrak{h}(\mathscr{A})$ assemble into a homomorphism of graded abelian groups, $\iota: \mathfrak{h}(A)^{\text {loc }} \rightarrow \mathfrak{h}(\mathscr{A})$. Let $\pi^{\prime}$ and $\iota^{\prime}$ be the restrictions of the aforementioned maps to derived Lie subalgebras. It is shown in [48, Prop. 2.1] that $\pi^{\prime} \circ \iota^{\prime}=\mathrm{id}$, and thus, $\pi^{\prime}$ is surjective (it is also injective in degree 2 ). We summarize this result, as follows.

## Proposition 7.2 ([48]). The morphism of graded Lie algebras

$$
\begin{equation*}
\mathfrak{h}\left(j_{\sharp}\right): \mathfrak{h}(\mathscr{A}) \longrightarrow \mathfrak{h}(\mathscr{A})^{\text {loc }} \tag{7.3}
\end{equation*}
$$

is a surjection in degrees $k \geqslant 3$ and an isomorphism in degree $k=2$.
By comparing the ranks of the source and target in (7.3), while also using (5.5) and (6.4), we recover a lower bound for the LCS ranks of an arrangement group, first established in [26, Prop. 3.8] by other methods.
Corollary 7.3 ([26,48]). For any arrangement $\mathscr{A}$, the LCS ranks of the group $G(\mathscr{A})$ admit the lower bound

$$
\begin{equation*}
\phi_{k}(G(\mathscr{A})) \geqslant \sum_{X \in L_{2}(\mathscr{A})} \phi_{k}\left(F_{\mu(X)}\right) \tag{7.4}
\end{equation*}
$$

for all $k \geqslant 2$, with equality for $k=2$.
As illustrated in the next example, the epimorphisms $\mathfrak{h}_{k}\left(j_{\sharp}\right)$ with $k \geqslant 3$ are far from being injective, in general. Thus, the lower bound from (7.4) may be strict, even for $k=3$.
Example 7.4. Let $\mathscr{B}$ be the braid arrangement in $\mathbb{C}^{3}$, defined by the polynomial $f=$ $(x+y)(x-y)(x+z)(x-z)(y+z)(y-z)$. Labeling the hyperplanes as the factors of $f$, the flats in $L_{2}(\mathscr{B})$ are $\{136,145,235,246,12,34,56\}$, and so $\mathfrak{h}_{3}(\mathscr{B})^{\text {loc }}=\mathbb{Z}^{8}$. Nevertheless,
$\mathfrak{h}_{3}(\mathscr{B})=\mathbb{Z}^{10}$, and thus $\operatorname{ker}\left(\mathfrak{h}_{3}\left(j_{\sharp}\right)\right)=\mathbb{Z}^{2}$, generated by the triple Lie brackets $\left[x_{2},\left[x_{4}, x_{5}\right]\right]$ and $\left[x_{5},\left[x_{2}, x_{4}\right]\right]$.
7.2. Decomposable arrangements. These considerations motivate the following definition, which is key to this work.
Definition 7.5 ([48]). A hyperplane arrangement $\mathscr{A}$ is said to be decomposable if the map $\mathfrak{h}_{3}\left(j_{\sharp}\right): \mathfrak{h}_{3}(\mathscr{A}) \rightarrow \mathfrak{h}_{3}\left(\mathscr{A}_{X}\right)^{\text {loc }}$ is an isomorphism. Likewise, the arrangement is decomposable over $\mathbb{Q}$ if the map $\mathfrak{h}_{3}\left(j_{\sharp}\right) \otimes \mathbb{Q}$ is an isomorphism.

That is to say, $\mathscr{A}$ is decomposable if $\mathfrak{h}_{3}(\mathscr{A})$ is free abelian of rank as small as possible, given the information encoded in $L_{2}(\mathscr{A})$, namely, of rank equal to

$$
\begin{equation*}
\sum_{X \in L_{2}(\mathscr{A})} \operatorname{rank} \mathfrak{h}_{3}\left(\mathscr{A}_{X}\right)=2 \sum_{X \in L_{2}(\mathscr{A})}\binom{\mu(X)+1}{3} \tag{7.5}
\end{equation*}
$$

Alternatively, if we set $\widetilde{L}_{2}(\mathscr{A}):=\left\{X \in L_{2}(\mathscr{A}): \mu(X)>1\right\}$, then the decomposability condition for $\mathscr{A}$ is equivalent to

$$
\begin{equation*}
\mathfrak{h}_{3}(\mathscr{A}) \cong \bigoplus_{X \in \tilde{L}_{2}(\mathscr{A})} \mathfrak{h}_{3}\left(\mathscr{A}_{X}\right) \tag{7.6}
\end{equation*}
$$

Since the holonomy Lie algebra $\mathfrak{h}(\mathscr{A})$ depends only on the intersection poset $L_{\leqslant 2}(\mathscr{A})$ the property of being decomposable (or $\mathbb{Q}$-decomposable) is combinatorially determined. We formalize this observation in the next lemma.

Lemma 7.6. Let $\mathscr{A}$ and $\mathscr{B}$ be two arrangement with $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathscr{B})$. If $\mathscr{A}$ is decomposable (over $\mathbb{Q}$ ), then $\mathscr{B}$ is also decomposable (over $\mathbb{Q}$ ).

The perhaps weaker condition of $\mathbb{Q}$-decomposability only requires that $\phi_{3}(G(\mathscr{A}))=$ $\operatorname{dim}_{\mathbb{Q}} \mathfrak{h}_{3}(\mathscr{A}) \otimes \mathbb{Q}$ be equal to (7.5), or, equivalently, that the lower bound from (7.4) to hold as equality for $k=3$. An alternate definition was given in [55, Definition 2.10], where an arrangement $\mathscr{A}$ is said to be minimal linear strand if

$$
\begin{equation*}
b_{2,3}^{\prime}(\mathscr{A})=2 \sum_{X \in L_{2}(\mathscr{A})}\binom{\mu(X)+1}{3} \tag{7.7}
\end{equation*}
$$

where $b_{i, j}^{\prime}:=\operatorname{dim}_{\mathbb{Q}} \operatorname{Tor}_{i}^{E}(A, \mathbb{Q})_{j}$ are the bigraded Betti numbers of the (rational) OrlikSolomon algebra $A=E / I(\mathscr{A})$. Since, as noted in display (3.13) of that paper, $\phi_{3}(G(\mathscr{A}))=$ $b_{2,3}^{\prime}(\mathscr{A})$, the two notions-decomposable over $\mathbb{Q}$ and minimal linear strand-coincide.

These considerations raise the following question.
Question 7.7. For an arrangement $\mathscr{A}$, are decomposability and $\mathbb{Q}$-decomposability equivalent conditions? Put another way: If $\mathscr{A}$ is decomposable over $\mathbb{Q}$, is $\mathfrak{h}_{3}(\mathscr{A})$ torsion-free?

Of course, if Question 6.3 has a positive answer for all arrangements $\mathscr{A}$ (that is, if $\mathfrak{h}_{3}(\mathscr{A})$ is always torsion-free), then Question 7.7 also has a positive answer, but the converse may or may not hold. An affirmative answer to Question 7.7 would make the algorithms for detecting decomposability (implemented in the Macaulay2 packages [19, 39], but only over a field of fixed characteristic) work over $\mathbb{Z}$.

The following theorem completely describes the structure of the holonomy Lie algebra and the associated graded Lie algebra of a decomposable arrangement. A similar proof, in a more abstract setting has since been given in [37] (see also [38]).

Theorem 7.8 ([48]). If $\mathscr{A}$ is a decomposable arrangement, then the following hold:
(1) The map $\mathfrak{h}^{\prime}\left(j_{\sharp}\right): \mathfrak{h}^{\prime}(\mathscr{A}) \rightarrow \prod_{X \in L_{2}(\mathscr{A})} \mathfrak{h}^{\prime}\left(\mathscr{A}_{X}\right)$ is an isomorphism of graded Lie algebras.
(2) The map $\Psi_{\mathscr{A}}: \mathfrak{h}(\mathscr{A}) \rightarrow \operatorname{gr}(G(\mathscr{A}))$ is an isomorphism of graded Lie algebras.

It follows from this theorem that $\mathfrak{h}_{n}(\mathscr{A}) \cong \operatorname{gr}_{n}(G(\mathscr{A}))$ for all $n \geqslant 1$, and all these groups are torsion-free, with ranks $\phi_{n}=\phi_{n}(G(\mathscr{A}))$ given by

$$
\begin{equation*}
\prod_{n=1}^{\infty}\left(1-t^{n}\right)^{\phi_{n}}=(1-t)^{|\mathscr{A}|-\sum_{X \in L_{2}(\mathscr{A})} \mu(X)} \prod_{X \in L_{2}(\mathscr{A})}(1-\mu(X) t) . \tag{7.8}
\end{equation*}
$$

Moreover, since the holonomy Lie algebra of any arrangement is combinatorially determined, we have the following immediate corollary.

Corollary 7.9. Let $\mathscr{A}$ and $\mathscr{B}$ be two decomposable arrangements with $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathscr{B})$. Then $\mathrm{gr}_{\geqslant 2}(G(\mathscr{A})) \cong \mathrm{gr}_{\geqslant 2}(G(\mathscr{B}))$.
7.3. Nilpotent quotients and localized arrangements. Building on the work from [48], we showed in [50] that the tower of nilpotent quotients of the fundamental group of the complement of a decomposable arrangement is fully determined by the intersection lattice. To explain this result, we start with some preparatory material on the second homology of these nilpotent groups. To start with, we have the following lemma, which is based on Theorems 4.3 and 7.8.

Lemma 7.10 ([50]). Let $\mathscr{A}$ be a decomposable arrangement, with complement $M=$ $M(\mathscr{A})$ and group $G=G(\mathscr{A})$. For every $k \geqslant 3$, there is a natural, split exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathfrak{h}_{k}(\mathscr{A}) \longrightarrow H_{2}\left(G / \gamma_{k}(G) ; \mathbb{Z}\right) \longrightarrow H_{2}(M ; \mathbb{Z}) \longrightarrow 0 \tag{7.9}
\end{equation*}
$$

For an arbitrary arrangement $\mathscr{A}$ and for a 2-flat $X \in L_{2}(\mathscr{A})$, we let $\mathscr{A}_{X}$ be the corresponding localized arrangement, and write $G_{X}=G\left(\mathscr{A}_{X}\right)$. The inclusion map $j_{X}: M(\mathscr{A}) \rightarrow$ $M\left(\mathscr{A}_{X}\right)$ induces a homomorphism $\left(j_{X}\right)_{\sharp}: G \rightarrow G_{X}$ on fundamental groups, which in turn
induces homomorphisms $N_{k}\left(j_{X}\right): G / \gamma_{n}(G) \rightarrow G_{X} / \gamma_{k}\left(G_{X}\right)$ on the respective nilpotent quotients. Assembling these maps, we obtain homomorphisms

$$
\begin{equation*}
N_{k}(j): G / \gamma_{k}(G) \longrightarrow \prod_{X \in L_{2}(\mathscr{A})} G_{X} / \gamma_{k}\left(G_{X}\right) \tag{7.10}
\end{equation*}
$$

for all $k \geqslant 1$.
Proposition 7.11 ([50]). For any arrangement $\mathscr{A}$, and for each $k \geqslant 3$, the map $N_{k}(j)$ induces a surjection in second homology,

$$
\begin{equation*}
N_{k}(j)_{*}: H_{2}\left(G / \gamma_{k}(G) ; \mathbb{Z}\right) \longrightarrow \oplus_{X \in L_{2}(\mathscr{A})} H_{2}\left(G_{X} / \gamma_{k}\left(G_{X}\right) ; \mathbb{Z}\right) . \tag{7.11}
\end{equation*}
$$

Moreover, if $\mathscr{A}$ is decomposable, then the maps $N_{k}(j)_{*}$ are isomorphisms, for all $k \geqslant 3$.
In [52,53] Rybnikov showed that, in general, the third nilpotent quotient of an arrangement group is not determined by the intersection lattice. Specifically, he produced a pair of arrangements $\mathscr{A}$ and $\mathscr{B}$, each one consisting of 13 hyperplanes in $\mathbb{C}^{3}$, such that $L(\mathscr{A}) \cong L(\mathscr{B})$, yet $G(\mathscr{A}) / \gamma_{4}(G(\mathscr{A})) \not \equiv G(\mathscr{B}) / \gamma_{4}(G(\mathscr{B}))$. In [3] Artal, Carmona, Cogolludo, and Marco gave a different proof of this result, based on a study of the truncations of the Alexander invariant.

By contrast, as shown in [50, Thm. 8.8], the phenomenon detected by Rybnikov cannot happen among decomposable arrangements.

Theorem 7.12 ([50]). Suppose $\mathscr{A}$ and $\mathscr{B}$ are decomposable arrangements with $L_{\leqslant 2}(\mathscr{A}) \cong$ $L_{\leqslant 2}(\mathscr{B})$. Then, for each $k \geqslant 2$, there is an isomorphism

$$
\begin{equation*}
G(\mathscr{A}) / \gamma_{k}(G(\mathscr{A})) \cong G(\mathscr{B}) / \gamma_{k}(G(\mathscr{B})) . \tag{7.12}
\end{equation*}
$$

In view of this theorem and of Lemma 7.6, all the nilpotent quotients of a decomposable arrangement group are combinatorially determined. This leaves open the following question.

Question 7.13. Is the group of a decomposable arrangement combinatorially determined?

Another open problem is whether decomposable arrangement groups are residually nilpotent. For more on this question, we refer to [13, §4.5] and [12] (see also Remark 8.7).

## 8. Constructions of decomposable arrangements

We now discuss some operations that preserve decomposability, and outline several methods of constructing decomposable arrangements.
8.1. Sub-arrangements and product arrangements. Decomposability behaves well with respect to some natural operations on arrangements. We start with a hereditary property, first established in [48, Prop. 3.3].

Proposition 8.1 ([48]). Let $\mathscr{A}$ be a decomposable arrangement (over $\mathbb{Q}$ ). If $\mathscr{B}$ is a subarrangement of $\mathscr{A}$, then $\mathscr{B}$ is also decomposable (over $\mathbb{Q}$ ).

This gives a convenient criterion for ruling out decomposability. For instance, if $\mathscr{A}$ is an arrangement that contains the braid arrangement $\mathscr{B}$ from Example 7.4 as a subarrangement, then, since $\mathscr{B}$ is not decomposable, $\mathscr{A}$ is not decomposable, either.

Next, we analyze the behavior of decomposability with respect to products. Let $\mathscr{A}$ and $\mathscr{B}$ be arrangements in $\mathbb{C}^{r}$ and $\mathbb{C}^{s}$, respectively. The product $\mathscr{A} \times \mathscr{B}$ defined in (6.6) is an arrangement in $\mathbb{C}^{r+s}$ with $M(\mathscr{A} \times \mathscr{B}) \cong M(\mathscr{A}) \times M(\mathscr{B})$. By (6.7), the ranked poset $L(\mathscr{A} \times \mathscr{B})$ is isomorphic to $L(\mathscr{A}) \times L(\mathscr{B})$; therefore, $\widetilde{L}_{2}(\mathscr{A} \times \mathscr{B})=\widetilde{L}_{2}(\mathscr{A}) \cup \widetilde{L}_{2}(\mathscr{B})$. Moreover, by (6.8), the graded Lie algebra $\mathfrak{h}(\mathscr{A} \times \mathscr{B})$ is isomorphic to $\mathfrak{h}(\mathscr{A}) \oplus \mathfrak{h}(\mathscr{B})$; in particular, $\mathfrak{h}_{3}(\mathscr{A} \times \mathscr{B}) \cong \mathfrak{h}_{3}(\mathscr{A}) \oplus \mathfrak{h}_{3}(\mathscr{B})$.

Proposition 8.2. Let $\mathscr{A}$ and $\mathscr{B}$ be two decomposable arrangements (over $\mathbb{Q}$ ). Then $\mathscr{A} \times \mathscr{B}$ is also decomposable (over $\mathbb{Q}$ ).

Proof. Suppose that both $\mathscr{A}$ and $\mathscr{B}$ are decomposable, that is, $\mathfrak{h}_{3}(\mathscr{A}) \cong \oplus_{X \in \tilde{L}_{2}(\mathscr{A})} \mathfrak{h}_{3}\left(\mathscr{A}_{X}\right)$ and $\mathfrak{h}_{3}(\mathscr{B}) \cong \bigoplus_{Y \in \tilde{L}_{2}(\mathscr{B})} \mathfrak{h}_{3}\left(\mathscr{B}_{Y}\right)$. Then

$$
\begin{align*}
\mathfrak{h}_{3}(\mathscr{A} \times \mathscr{B}) & \cong \mathfrak{h}_{3}(\mathscr{A}) \oplus \mathfrak{h}_{3}(\mathscr{B}) \\
& \cong\left(\bigoplus_{X \in \tilde{L}_{2}(\mathscr{A})} \mathfrak{h}_{3}\left(\mathscr{A}_{X}\right)\right) \oplus\left(\bigoplus_{Y \in \tilde{L}_{2}(\mathscr{B})} \mathfrak{h}_{3}\left(\mathscr{B}_{Y}\right)\right)  \tag{8.1}\\
& \cong \bigoplus_{Z \in \tilde{L}_{2}(\mathscr{A} \times \mathscr{B})} \mathfrak{h}_{3}\left((\mathscr{A} \times \mathscr{B})_{Z}\right),
\end{align*}
$$

and so $\mathscr{A} \times \mathscr{B}$ is decomposable. The case when $\mathscr{A}$ and $\mathscr{B}$ are decomposable over $\mathbb{Q}$ is treated in the same manner.
8.2. Pencils in general position. Let $m=\left(m_{1}, \ldots, m_{r}\right)$ be an $r$-tuple of integers with $m_{i} \geqslant 2$, and let $\hat{\mathscr{A}}(m)$ be the arrangement hyperplanes in $\mathbb{C}^{r+1}$ defined by the polynomial

$$
\begin{equation*}
f\left(z_{0}, z_{1}, \ldots, z_{r}\right)=z_{0}\left(z_{0}^{m_{1}}-z_{1}^{m_{1}}\right)\left(z_{0}^{m_{2}}-z_{2}^{m_{2}}\right) \cdots\left(z_{0}^{m_{r}}-z_{r}^{m_{r}}\right) . \tag{8.2}
\end{equation*}
$$

Clearly, the projectivized complement, $U(\hat{\mathscr{A}}(m))$, is homeomorphic to $\prod_{i=1}^{r}\left(\mathbb{C} \backslash\left\{m_{i}\right.\right.$ points $\left.\}\right)$.
Now let $\mathscr{A}(m)$ be a generic 3-slice of this arrangement (said to be "split-solvable" in [9]). Its projectivization, $\overline{\mathscr{A}}(m)$, is an arrangement of lines in $\mathbb{C P}^{2}$, consisting of $r$ pencils meeting the line at infinity, $z_{0}=0$, in points with multiplicities $m_{1}+1, \ldots, m_{r}+1$; moreover, the lines of any two distinct pencils are in general position with each other. In an affine
chart, $\overline{\mathscr{A}}(m)$ consists of $r$ packets of $m_{i}$ parallel lines, with the lines in distinct packets having different slopes.

Lemma 8.3. The arrangements $\mathscr{A}(m)$ are decomposable.

Proof. From the above description of the line arrangement $\overline{\mathscr{A}}(m)$, we see that $\widetilde{L}_{2}(\mathscr{A}(m))=$ $\left\{X_{1}, \ldots, X_{r}\right\}$, where $\mu\left(X_{i}\right)=m_{i}$. On the other hand, by a Lefschetz-type theorem of Hamm and Lê, the group $\pi_{1}(U(\mathscr{A}(m)))$ is isomorphic to $\pi_{1}(U(\hat{\mathscr{A}}(m)))$, which, by a previous observation, is isomorphic to $F_{m_{1}} \times \cdots \times F_{m_{r}}$. It follows that $\mathfrak{h}(\mathscr{A}(m))=\mathfrak{h}\left(\pi_{1}(M(\mathscr{A}(m)))\right.$ is isomorphic to $\operatorname{Lie}(1) \times \operatorname{Lie}\left(m_{1}\right) \times \cdots \times \operatorname{Lie}\left(m_{r}\right)$, from which we conclude that $\mathfrak{h}_{3}(\mathscr{A}(m))=$ $\oplus_{i=1}^{r} \mathfrak{h}_{3}\left(\mathscr{A}(m)_{X_{i}}\right)$, thus showing that $\mathscr{A}(m)$ is decomposable.

Since the decomposability of an arrangement $\mathscr{A}$ depends only on the (truncated) intersection lattice $L_{\leqslant 2}(\mathscr{A})$, we obtain the following immediate corollary.

Corollary 8.4. Let $\mathscr{A}$ be an arrangement such that $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathscr{A}(m))$, for some $r$-tuple $m=\left(m_{1}, \ldots, m_{r}\right)$ with $m_{i} \geqslant 2$. Then $\mathscr{A}$ is decomposable.

Using the work of Jiang and Yau $[31,32]$ relating the topology and combinatorics of line arrangements with "nice" intersection posets, Choudary, Dimca, and Papadima proved in [9, Corollary 1.7] the following result, which we shall need later on.

Theorem 8.5 ([9]). Let $\mathscr{A}$ be a central arrangement in $\mathbb{C}^{3}$. The following are equivalent:
(1) $L(\mathscr{A}) \cong L(\mathscr{A}(m))$, for some $r$-tuple $m=\left(m_{1}, \ldots, m_{r}\right)$.
(2) $U(\mathscr{A}) \cong U(\mathscr{A}(m))$.
(3) $\pi_{1}(U(\mathscr{A}))$ is isomorphic to $F_{m_{1}} \times \cdots \times F_{m_{r}}$ via an isomorphism preserving the standard generators in $H_{1}$.
8.3. Decomposable graphic arrangements. Let $\Gamma=(\mathrm{V}, \mathrm{E})$ be a finite simplicial graph with vertex set $\mathrm{V}=[n]:=\{1, \ldots, n\}$ and edge set $\mathrm{E} \subset 2^{[n]}$. To such a graph there corresponds a graphic arrangement, denoted by $\mathscr{A}_{\Gamma}$, which consists of the hyperplanes $H_{e}=\left\{z_{i}-z_{j}=0\right\}$ in $\mathbb{C}^{n}$ indexed by the edges $e=\{i, j\}$ of E . For example, if $\Gamma=K_{n}$, the complete graph on $n$ vertices, then $\mathscr{A}_{n}:=\mathscr{A}_{K_{n}}$ is the braid arrangement in $\mathbb{C}^{n}$ from Section 5.4. Thus, any graphic arrangement $\mathscr{A}_{\Gamma}$ can be viewed as a sub-arrangement of the braid arrangement $\mathscr{A}_{n}$, where $n=|\mathrm{V}|$.

For each 2-flat $X \in L_{2}\left(\mathscr{A}_{\Gamma}\right)$, there are either 2 or 3 hyperplanes containing $X$. Under the bijection between $\mathscr{A}_{\Gamma}$ and E sending $H_{e}$ to $e$, a flat of size 3 of corresponds to a triangle in the graph, while a flat of size 2 corresponds to a pair of edges which is not included in any element of the triangle-set T. Therefore, the holonomy Lie algebra $\mathfrak{h}(\Gamma):=\mathfrak{h}\left(G\left(\mathscr{A}_{\Gamma}\right)\right)$ is the quotient of the free Lie algebra on variables $e \in \mathrm{E}$ by the corresponding ideal of quadratic
relations,

$$
\mathfrak{h}(\Gamma)=\operatorname{Lie}(\mathrm{E}) / \operatorname{ideal}\left\{\begin{array}{ll}
{\left[e_{1}, e_{2}+e_{3}\right],} & \text { if }\left\{e_{1}, e_{2}, e_{3}\right\} \in \mathrm{T}  \tag{8.3}\\
{\left[e_{1}, e_{2}\right],} & \text { if }\left\{e_{1}, e_{2}, e\right\} \notin \mathrm{T}, \forall e \in \mathrm{E}
\end{array}\right\} .
$$

Proposition 8.6 ([48]). For a graphic arrangement $\mathscr{A}_{\Gamma}$, the following conditions are equivalent.
(1) $\mathscr{A}_{\Gamma}$ is decomposable.
(2) $\mathscr{A}_{\Gamma}$ is decomposable over $\mathbb{Q}$.
(3) $\Gamma$ contains no complete subgraphs on 4 vertices.

Let $\kappa_{s}=\kappa_{s}(\Gamma)$ be the number of $K_{s+1}$ subgraphs of $\Gamma$, so that $\kappa_{0}=|\mathrm{V}|, \kappa_{1}=|\mathrm{E}|$, and $\kappa_{2}=|\mathrm{T}|$. If $\Gamma$ contains no $K_{4}$ subgraphs, i.e., $\kappa_{3}=0$, then the above proposition together with formula (7.8) show that the LCS ranks $\phi_{k}=\phi_{k}\left(G\left(\mathscr{A}_{\Gamma}\right)\right)$ are given by

$$
\begin{equation*}
\prod_{k=1}^{\infty}\left(1-t^{k}\right)^{\phi_{k}}=(1-t)^{\kappa_{1}-2 \kappa_{2}}(1-2 t)^{\kappa_{2}} . \tag{8.4}
\end{equation*}
$$

In fact, as conjectured in [55] and as proved in [36], the following LCS formula holds for any graphic arrangement in $\mathbb{C}^{n}$,

$$
\begin{equation*}
\prod_{k=1}^{\infty}\left(1-t^{k}\right)^{\phi_{k}}=\prod_{j=1}^{n-1}(1-j t)^{\sum_{s=j}^{n-1}(-1)^{s-j}\binom{s}{j}^{\kappa_{s}}} \tag{8.5}
\end{equation*}
$$

or, equivalently, $\phi_{k}=\sum_{j=1}^{k} \sum_{s=j}^{k}(-1)^{s-j}\binom{s}{j} \kappa_{s} \phi_{k}\left(F_{j}\right)$.
Remark 8.7. In [12, Thm. 5.5], Cohen and Falk show the following: If $\Gamma$ is $K_{4}$-free, then the graphic arrangement group $P_{\Gamma}=\pi_{1}\left(G\left(\mathscr{A}_{\Gamma}\right)\right)$ embeds as a subgroup of a finite direct product of pure braid groups, and thus is residually (torsion-free) nilpotent. It would be interesting to know whether $P_{\Gamma} / P_{\Gamma}^{\prime \prime}$ is also residually nilpotent.
8.4. More decomposable arrangements. We conclude this section with three more examples of decomposable arrangements that are not of the aforementioned types. More details about these well-known arrangements (the $\mathrm{X}_{3}, \mathrm{X}_{2}$, and non-Pappus arrangements) can be found in [57].

The first is the $\mathrm{X}_{3}$ arrangement, with defining polynomial $f=x y z(x+y)(x+z)(y+z)$. Ordering the hyperplanes of $\mathscr{A}$ as the linear factors of $f$, the 2-flats in $\widetilde{L}_{2}(\mathscr{A})$ are $\{1,2,4\}$, $\{1,3,5\}$, and $\{2,3,6\}$. The holonomy Lie algebra $\mathfrak{h}(\mathscr{A})$ has degree 3 piece isomorphic to $\mathfrak{h}_{3}^{\text {loc }}(\mathscr{A}) \cong \mathbb{Z}^{6}$, with basis $\left[x_{1},\left[x_{1}, x_{4}\right]\right],\left[x_{4},\left[x_{1}, x_{4}\right]\right],\left[x_{1},\left[x_{1}, x_{5}\right]\right],\left[x_{5},\left[x_{1}, x_{5}\right]\right],\left[x_{2},\left[x_{2}, x_{6}\right]\right]$, $\left[x_{6},\left[x_{2}, x_{6}\right]\right]$. Therefore, $\mathscr{A}$ is decomposable, and so the group $G(\mathscr{A})$ has the same LCS ranks as the group $F_{2}^{\times 3}=F_{2} \times F_{2} \times F_{2}$. Nevertheless, $G(\mathscr{A})$ is not isomorphic to a finite direct product of finitely generated free groups. Indeed, this arrangement group is isomorphic to the celebrated Stallings group (the kernel of the epimorphism $F_{2}^{\times 3} \rightarrow \mathbb{Z}$ which
sends each standard generator to 1 ), see [59, Rem. 12.4]. It follows that $H_{3}(G(\mathscr{A}) ; \mathbb{Z})$ is not finitely generated, and thus $G(\mathscr{A})$ is of type $F_{2}$ (finitely presented), but not of type $F_{3}$ (it does not admit a classifying space with finite 3-skeleton).

The next example is the $\mathrm{X}_{2}$ arrangement, with defining polynomial $f=x y z(y-z)(x-$ $z)(x+y)(x+y-2 z)$. The 2-flats in $\widetilde{L}_{2}(\mathscr{A})$ are $\{1,2,6\},\{1,3,5\},\{2,3,4\},\{3,6,7\}$, and $\{4,5,7\}$. Once again, $\mathfrak{h}_{3}(\mathscr{A})$ is isomorphic to $\mathfrak{h}_{3}^{\text {loc }}(\mathscr{A}) \cong \mathbb{Z}^{10}$, and so $\mathscr{A}$ is decomposable. Therefore, $\phi_{k}(G(\mathscr{A}))=\phi_{k}\left(F_{2}^{\times 5}\right)$ for all $k \geqslant 2$, yet clearly $G(\mathscr{A})$ is not isomorphic to a direct product of finitely generated free groups, since $b_{1}(G(\mathscr{A}))=7<10$.

Finally, let $\mathscr{A}$ be the non-Pappus arrangement; this is a realization of the $\left(9_{3}\right)_{2}$ configuration of Hilbert and Cohn-Vossen, with defining polynomial $f=x y z(x+y)(y+z)(x+$ $3 z)(x+2 y+z)(x+2 y+3 z)(2 x+3 y+3 z)$. There are 9 flats in $\widetilde{L}_{2}(\mathscr{A}):\{1,2,4\},\{1,3,6\}$, $\{1,5,9\},\{2,3,5\},\{2,6,8\},\{3,7,8\},\{4,5,7\},\{4,8,9\}$, and $\{6,7,9\}$. It is readily checked that $\mathfrak{h}_{3}(\mathscr{A})$ is isomorphic to $\mathfrak{h}_{3}^{\text {loc }}(\mathscr{A}) \cong \mathbb{Z}^{18}$, and thus $\mathscr{A}$ is decomposable; nevertheless, $G(\mathscr{A})$ is not a direct product of finitely generated free groups,

## 9. Alexander invariants and Chen ranks of arrangements

In this section we study the Alexander invariants of hyperplane arrangements, their relation to the "local" Alexander invariants, and how all this informs on the Chen ranks of arrangement groups.
9.1. Alexander invariants of arrangements. Let $\mathscr{A}$ be a complex hyperplane arrangement, with complement $M(\mathscr{A})$ and group $G=G(\mathscr{A})=\pi_{1}(M(\mathscr{A}))$. Following [15, 17] (see also [3]), we define the Alexander invariant of $\mathscr{A}$ as the Alexander invariant of its fundamental group,

$$
\begin{equation*}
B(\mathscr{A}):=B(G(\mathscr{A}))=G^{\prime} / G^{\prime \prime}, \tag{9.1}
\end{equation*}
$$

viewed as a module over the group ring $R=\mathbb{Z}\left[G_{\mathrm{ab}}\right]=\mathbb{Z}\left[H_{1}(M(\mathscr{A}) ; \mathbb{Z})\right]$. Upon choosing an ordering $\mathscr{A}=\left\{H_{1}, \ldots, H_{n}\right\}$ on the hyperplanes, we may identify $G_{\text {ab }}$ with the free abelian group on the meridians $\left\{x_{1}, \ldots, x_{n}\right\}$, and the ring $R$ with the ring of Laurent polynomials $\mathbb{Z}\left[t_{1}^{ \pm 1}, \ldots, t_{n}^{ \pm 1}\right]$.

The Alexander invariant $B=B(\mathscr{A})$ comes equipped with a filtration by the powers of the augmentation ideal, $I=\operatorname{ker}(\varepsilon: R \rightarrow \mathbb{Z})$. The $I$-adic completion of the Alexander invariant, $\widehat{B}$, is a module over the ring $\widehat{R}$, filtered by the powers of the ideal $\hat{I}$. Moreover, the associated graded modules, $\operatorname{gr}(B)$ and $\operatorname{gr}(\widehat{B})$, are isomorphic as modules over the graded ring $\operatorname{gr}(R)=\operatorname{gr}(\widehat{R})$, which may be identified with the polynomial ring $S=\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$, with variables $x_{i}$ in degree 1 corresponding to the elements $t_{i}-1 \in I$.

We do not know of any arrangement $\mathscr{A}$ for which the Alexander invariant $B(\mathscr{A})$ is not separated in the $I$-adic topology, but we suspect that additional assumptions (such as
decomposability) are needed in order to insure separability. We shall come back to this topic in Question 10.10 (see also Example 12.4).
9.2. From global to local Alexander invariants. For each 2-flat $X \in L_{2}(\mathscr{A})$, we have a "local" Alexander invariant, $B\left(\mathscr{A}_{X}\right)$, viewed as a module over the group ring $R_{X}=$ $\mathbb{Z}\left[H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)\right]$. As noted previously, $G\left(\mathscr{A}_{X}\right) \cong F_{\mu(X)} \times \mathbb{Z}$; thus, by Corollary 2.13, the module $B\left(\mathscr{A}_{X}\right) \cong B\left(F_{\mu(X)} \times \mathbb{Z}\right)$ is separated in the $I_{X}$-adic topology, where $I_{X}$ is the augmentation ideal of $R_{X}$.

Recall that the inclusion $j^{X}: M(\mathscr{A}) \hookrightarrow M\left(\mathscr{A}_{X}\right)$ induces a surjective homomorphism $j_{\sharp}^{X}: G(\mathscr{A}) \rightarrow G\left(\mathscr{A}_{X}\right)$. Therefore, we get a surjective morphism

$$
\begin{equation*}
B\left(j_{\sharp}^{X}\right): B(\mathscr{A}) \longrightarrow B\left(\mathscr{A}_{X}\right) \tag{9.2}
\end{equation*}
$$

between the respective Alexander invariants, which covers the ring map $\tilde{j}_{*}^{X}: R \rightarrow R_{X}$ induced by the epimorphism $j_{*}^{X}: H_{1}(M(\mathscr{A}) ; \mathbb{Z}) \rightarrow H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$.

Now let $j_{\sharp}: G(\mathscr{A}) \rightarrow \prod_{X \in L_{2}(\mathscr{A})} G\left(\mathscr{A}_{X}\right)$ be the homomorphism induced on fundamental groups by the map $j: M(\mathscr{A}) \rightarrow \prod_{X \in L_{2}(\mathscr{A})} M\left(\mathscr{A}_{X}\right)$ from (7.1). The abelianization of this homomorphism coincides with the induced homomorphism $j_{*}: H_{1}(M(\mathscr{A}) ; \mathbb{Z}) \rightarrow$ $\oplus_{X \in L_{2}(\mathscr{A})} H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$, which was shown in Lemma 7.1 to be injective. Therefore, by Lemma 2.9, the extension of the map $j_{*}=\left(j_{\sharp}\right)_{\text {ab }}$ to group rings,

$$
\begin{equation*}
\tilde{j}_{*}=\left(\tilde{j}_{*}^{X}\right)_{X}: \mathbb{Z}\left[H_{1}(M(\mathscr{A}) ; \mathbb{Z})\right] \longrightarrow \prod_{X \in L_{2}(\mathscr{A})} \mathbb{Z}\left[H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)\right] . \tag{9.3}
\end{equation*}
$$

is also injective. The maps $B\left(j_{\sharp}^{X}\right)$ from (9.2) assemble into a morphism

$$
\begin{equation*}
B\left(j_{\sharp}\right)=\left(B\left(j_{\sharp}^{X}\right)\right)_{X}: B(\mathscr{A}) \longrightarrow \oplus_{X \in L_{2}(\mathscr{A})} B\left(\mathscr{A}_{X}\right) \tag{9.4}
\end{equation*}
$$

which covers the injective ring map $\tilde{j}_{*}: R \rightarrow \prod_{X} R_{X}$ from (9.3). Let

$$
\begin{equation*}
B(\mathscr{A})^{\mathrm{loc}}:=\left(\bigoplus_{X \in L_{2}(\mathscr{A})} B\left(\mathscr{A}_{X}\right)\right)_{j_{\sharp}}=\bigoplus_{X \in L_{2}(\mathscr{A})} B\left(\mathscr{A}_{X}\right)_{j_{\sharp}^{X}} \tag{9.5}
\end{equation*}
$$

be the $R$-module obtained from $\bigoplus_{X} B\left(\mathscr{A}_{X}\right)$ by restriction of scalars along the map $\tilde{j}_{*}$. This module was first considered in [17], where it was called the coarse combinatorial Alexander invariant of $\mathscr{A}$. By Remark 2.1, the map $B\left(j_{\sharp}\right)$ from (9.4) factors as

$$
\begin{equation*}
B(\mathscr{A}) \xrightarrow{\Pi} B(\mathscr{A})^{\mathrm{loc}} \longrightarrow \oplus_{X \in L_{2}(\mathscr{A})} B\left(\mathscr{A}_{X}\right), \tag{9.6}
\end{equation*}
$$

where the first arrow is an $R$-linear map and the second arrow is the identity map of $\oplus_{X} B\left(\mathscr{A}_{X}\right)$, viewed as covering the ring map $\tilde{j}_{*}$.
9.3. Infinitesimal Alexander invariants of arrangements. Once again, let $\mathscr{A}$ be a hyperplane arrangement, with group $G(\mathscr{A})$ and holonomy Lie algebra $\mathfrak{h}(\mathscr{A})=\mathfrak{h}(G(\mathscr{A}))$. We define the infinitesimal Alexander invariant of $\mathscr{A}$ as

$$
\begin{equation*}
\mathfrak{B}(\mathscr{A}):=\mathfrak{B}(G(\mathscr{A}))=\mathfrak{h}^{\prime}(\mathscr{A}) / \mathfrak{h}^{\prime \prime}(\mathscr{A}), \tag{9.7}
\end{equation*}
$$

viewed as a graded module over the symmetric algebra $S=\operatorname{Sym}\left[G_{\mathrm{ab}}\right]$. Since $G_{\mathrm{ab}}=$ $H_{1}(M(\mathscr{A}) ; \mathbb{Z})$ and $R=\mathbb{Z}\left[H_{1}(M(\mathscr{A}) ; \mathbb{Z})\right]$, the ring $S$ is isomorphic (as a graded ring) to $\operatorname{gr}(R)$. As noted previously, fixing an ordering $H_{1}, \ldots, H_{n}$ of the hyperplanes in $\mathscr{A}$ yields an isomorphism $S \cong \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$.

Recall that the arrangement group $G(\mathscr{A})$ is 1-formal. Thus, by Corollary 4.6, the $S \otimes \mathbb{Q}$ module $\mathfrak{B}(\mathscr{A}) \otimes \mathbb{Q}$ is isomorphic to $\operatorname{gr}(B(\mathscr{A})) \otimes \mathbb{Q}$.

To each flat $X \in L_{2}(\mathscr{A})$ there corresponds a "local" infinitesimal Alexander invariant, $\mathfrak{B}\left(\mathscr{A}_{X}\right)$, viewed as a module over the polynomial ring $S_{X}=\operatorname{Sym}\left[H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)\right] \cong$ $\operatorname{gr}\left(R_{X}\right)$. Recall that the inclusion $j^{X}: M(\mathscr{A}) \hookrightarrow M\left(\mathscr{A}_{X}\right)$ induces a surjective homomorphism $j_{\sharp}^{X}: G(\mathscr{A}) \rightarrow G\left(\mathscr{A}_{X}\right)$. Therefore, we get an epimorphism $\mathfrak{B}\left(j_{\sharp}^{X}\right): \mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}\left(\mathscr{A}_{X}\right)$ that covers the ring map $\tilde{j}_{*}^{X}: S \rightarrow S_{X}$ obtained by extending to symmetric algebras the homomorphism $j_{*}^{X}: H_{1}(M(\mathscr{A}) ; \mathbb{Z}) \rightarrow H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)$.

The maps $\mathfrak{B}\left(j_{\sharp}^{X}\right)$ assemble into a morphism $\mathfrak{B}\left(j_{\sharp}\right)=\left(\mathfrak{B}\left(j_{\sharp}^{X}\right)\right)_{X}$ from $\mathfrak{B}(\mathscr{A})$ to $\oplus_{X} \mathfrak{B}\left(\mathscr{A}_{X}\right)$ that covers the ring map $\bar{j}_{*}: S \rightarrow \prod_{X} S_{X}$ corresponding to $\operatorname{gr}\left(\tilde{j}_{*}\right): \operatorname{gr}(R) \rightarrow \prod_{X} \operatorname{gr}\left(R_{X}\right)$. By Lemmas 2.9 and 7.1, this ring map is injective. Let

$$
\begin{equation*}
\mathfrak{B}(\mathscr{A})^{\mathrm{loc}}:=\left(\bigoplus_{X \in L_{2}(\mathscr{A})} \mathfrak{B}\left(\mathscr{A}_{X}\right)\right)_{j_{\sharp}}=\bigoplus_{X \in L_{2}(\mathscr{A})} \mathfrak{B}\left(\mathscr{A}_{X}\right)_{j_{\sharp}^{X}} \tag{9.8}
\end{equation*}
$$

be the $S$-module obtained from $\oplus_{X} \mathfrak{B}\left(\mathscr{A}_{X}\right)$ by restriction of scalars along the map $\operatorname{gr}\left(\tilde{j}_{*}\right)$. By Remark 2.1, the map $\mathfrak{B}\left(j_{\sharp}\right)$ may be viewed as the composite

$$
\begin{equation*}
\mathfrak{B}(\mathscr{A}) \xrightarrow{\bar{\Pi}} \mathfrak{B}(\mathscr{A})^{\mathrm{loc}} \longrightarrow \oplus_{X \in L_{2}(\mathscr{A})} \mathfrak{B}\left(\mathscr{A}_{X}\right), \tag{9.9}
\end{equation*}
$$

where the first arrow is an $S$-linear map and the second arrow is the identity map of $\oplus_{X} \mathfrak{B}\left(\mathscr{A}_{X}\right)$, viewed as covering the ring map $\bar{j}_{*}$.
9.4. Bounding the Alexander invariants arrangements. We are now ready to state and prove the main result of this section.

Theorem 9.1. For any arrangement $\mathscr{A}$, the following hold.
(1) The morphism $\bar{\Pi}: \mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}(\mathscr{A})^{\text {loc }}$ is surjective.
(2) The morphism $\widehat{\Pi} \otimes \mathbb{Q}: \widehat{B(\mathscr{A})} \otimes \mathbb{Q} \rightarrow \widehat{B(\mathscr{A})^{\text {loc }}} \otimes \mathbb{Q}$ is surjective.
(3) The morphism $\Pi \otimes \mathbb{Q}: B(\mathscr{A}) \otimes \mathbb{Q} \rightarrow B(\mathscr{A})^{\mathrm{loc}} \otimes \mathbb{Q}$ is surjective.

Proof. Let $j: M(\mathscr{A}) \rightarrow \prod_{X \in L_{2}(\mathscr{A})} M\left(\mathscr{A}_{X}\right)$ be the map from (7.1). The corresponding homomorphism on fundamental groups, $j_{\sharp}: G(\mathscr{A}) \rightarrow \prod_{X} G\left(\mathscr{A}_{X}\right)$, induces a map of graded Lie algebras,

$$
\begin{equation*}
\mathfrak{h}\left(j_{\sharp}\right): \mathfrak{h}(\mathscr{A}) \longrightarrow \mathfrak{h}\left(\prod_{X} \mathscr{A}_{X}\right) \cong \prod_{X} \mathfrak{h}\left(\mathscr{A}_{X}\right), \tag{9.10}
\end{equation*}
$$

where the isomorphism on the right is the one provided by Lemma 4.1. By Proposition 7.2, the restriction $\mathfrak{h}^{\prime}\left(j_{\sharp}\right): \mathfrak{h}^{\prime}(\mathscr{A}) \rightarrow \prod_{X} \mathfrak{h}^{\prime}\left(\mathscr{A}_{X}\right)$ is surjective. This map induces a map on quotients,

$$
\begin{equation*}
\mathfrak{h}^{\prime}(\mathscr{A}) / \mathfrak{h}^{\prime \prime}(\mathscr{A}) \longrightarrow \prod_{X \in L_{2}(\mathscr{A})} \mathfrak{h}^{\prime}\left(\mathscr{A}_{X}\right) / \mathfrak{h}^{\prime \prime}\left(\mathscr{A}_{X}\right), \tag{9.11}
\end{equation*}
$$

which also must be surjective. Observe that the map (9.11) coincides with the composite map from (9.9). Therefore, the $S$-morphism $\bar{\Pi}: \mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}(\mathscr{A})^{\text {loc }}$ is surjective, and this proves claim (1).

Let us pass now to completions. Since arrangement groups are 1-formal, Theorem 4.5 gives a filtration-preserving isomorphism of $\widehat{R} \otimes \mathbb{Q}$-modules, $\Phi_{\mathscr{A}}: \widehat{B(\mathscr{A})} \otimes \mathbb{Q} \xrightarrow{\simeq} \widehat{\mathfrak{B}(\mathscr{A})} \otimes \mathbb{Q}$, and, for each $X \in L_{2}(\mathscr{A})$, an isomorphism of $\widehat{R}_{X} \otimes \mathbb{Q}$-modules, $\Phi_{\mathscr{A}_{X}}: \widehat{B\left(\mathscr{A}_{X}\right)} \otimes \mathbb{Q} \xrightarrow{\simeq}$ $\widehat{\mathfrak{B}\left(\mathscr{A}_{X}\right)} \otimes \mathbb{Q}$. Furthermore, these isomorphisms are compatible; indeed, by (4.8), we have a commutative diagram,
with horizontal arrows covering the ring map $\widehat{\tilde{j}}_{*}^{X}: \widehat{R} \rightarrow \widehat{R}_{X}$. Taking direct sums over all the 2-flats of $\mathscr{A}$, and letting $\Phi_{\mathscr{A}}^{\text {loc }}:=\oplus_{X \in L_{2}(\mathscr{A})} \Phi_{\mathscr{\mathscr { I } _ { X }}}$, we obtain the following commutative diagram in the category of $\widehat{R} \otimes \mathbb{Q}$-modules,


By part (1) and Lemma 2.15, the bottom arrow in this diagram is surjective. Thus, the top arrow is also surjective, and this proves claim (2).

Finally, using again Lemma 2.15, it follows from part (2) that the map $\Pi \otimes \mathbb{Q}$ is an epimorphism of $R \otimes \mathbb{Q}$-modules; this proves claim (3) and completes the proof.

Remark 9.2. An explicit description of the $R$-morphism $\Pi: B(\mathscr{A}) \rightarrow B(\mathscr{A})^{\text {loc }}$ was given in [17, Thm 6.3]. This description involves certain presentations for the respective Alexander invariants, obtained by means of the Fox Calculus applied to the braid monodromy presentations of the groups $G(\mathscr{A})$ and $G\left(\mathscr{A}_{X}\right)$. That approach (which requires considerably more work), shows that, in fact, the map $\Pi: B(\mathscr{A}) \rightarrow B(\mathscr{A})^{\text {loc }}$ itself is surjective, not just the rationalization $\Pi \otimes \mathbb{Q}$ from Theorem 9.1, part (3). Moreover, it was shown in [17, Cor. 6.6] that the $\widehat{R}$-module $\widehat{B(\mathscr{A})}$ admits an (explicit) presentation with $\binom{b_{1}}{2}-b_{2}$ generators and $\binom{b_{1}}{3}$ relations, where the Betti numbers $b_{k}=b_{k}(M(\mathscr{A}))$ are given by (5.4). No such presentation is known in general for either $B(\mathscr{A})$ or $\operatorname{gr}(B(\mathscr{A}))$.
9.5. Bounding the Chen ranks of arrangements. The study of the Alexander invariants of arrangement groups undertaken in [17] led to lower bounds for the Chen ranks of those groups. Notably, those bounds were expressed solely in terms of the Möbius function of the intersection lattice of the arrangement. We present here a different approach to obtaining those bounds, without appealing to explicit presentations for the (completed) Alexander invariants of arrangement groups.

To start with, recall that arrangement groups are 1 -formal. Hence, as a direct consequence of Corollary 4.7, we obtain the following result, which is the content of [47, Thm. 11.1].

Corollary 9.3 ([47]). The generating series for the Chen ranks of an arrangement group $G(\mathscr{A})$ is given by

$$
\begin{equation*}
\sum_{n \geqslant 0} \theta_{k+2}(G(\mathscr{A})) t^{k}=\operatorname{Hilb}(\mathfrak{B}(\mathscr{A}) \otimes \mathbb{Q}, t) . \tag{9.14}
\end{equation*}
$$

Consequently, the Chen ranks of $G(\mathscr{A})$ are determined by $L_{\leqslant 2}(\mathscr{A})$.

To each flat $X \in L_{2}(\mathscr{A})$, there corresponds a "local" infinitesimal Alexander invariant, $\mathfrak{B}\left(\mathscr{A}_{X}\right)$, viewed as a module over the polynomial ring $S_{X}=\operatorname{Sym}\left[H_{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{Z}\right)\right] \cong$ $\operatorname{gr}\left(R_{X}\right)$. Recall from (5.5) that the group $G\left(\mathscr{A}_{X}\right)$ is isomorphic to $F_{\mu(X)} \times \mathbb{Z}$; in particular, $\theta_{1}\left(G\left(\mathscr{A}_{X}\right)\right)=\mu(X)+1$. For $k \geqslant 2$, Corollary 9.3 , together with the computations from Examples 2.3 and 3.6, yields the equalities

$$
\begin{equation*}
\theta_{k}\left(G\left(\mathscr{A}_{X}\right)\right)=\operatorname{dim}_{\mathbb{Q}}\left(\mathfrak{B}_{k-2}\left(\mathscr{A}_{X}\right) \otimes \mathbb{Q}\right)=(k-1)\binom{\mu(X)+k-2}{k} . \tag{9.15}
\end{equation*}
$$

Note that if $\mu(X)=1$, then $G\left(\mathscr{A}_{X}\right) \cong \mathbb{Z}^{2}$ and $\theta_{k}\left(G\left(\mathscr{A}_{X}\right)\right)=0$ for all $k \geqslant 2$.
The next result recovers [17, Cor. 7.2] using our alternative approach, based on infinitesimal Alexander invariants.

Corollary 9.4 ([17]). The Chen ranks of a hyperplane arrangement $\mathscr{A}$ admit the lower bound

$$
\begin{equation*}
\theta_{k}(G(\mathscr{A})) \geqslant \sum_{X \in \tilde{L}_{2}(\mathscr{A})} \theta_{k}\left(G\left(\mathscr{A}_{X}\right)\right)=(k-1) \sum_{X \in \tilde{L}_{2}(\mathscr{A})}\binom{\mu(X)+k-2}{k}, \tag{9.16}
\end{equation*}
$$

valid for all $k \geqslant 2$, with equality for $k=2$.
Proof. Fix an integer $k \geqslant 2$. Then,

$$
\begin{align*}
\theta_{k}(G(\mathscr{A})) & =\operatorname{dim}_{\mathbb{Q}} \mathfrak{B}_{k-2}(\mathscr{A}) \otimes \mathbb{Q} & & \text { by Corollary 9.3 } \\
& \geqslant \operatorname{dim}_{\mathbb{Q}} \mathfrak{B}_{k-2}(\mathscr{A})^{\text {loc }} \otimes \mathbb{Q} & & \text { by Theorem 9.1, part (1) } \\
& =\sum_{X \in L_{2}(\mathscr{A})} \operatorname{dim}_{\mathbb{Q}} \mathfrak{B}_{k-2}\left(\mathscr{A}_{X}\right) \otimes \mathbb{Q} & & \text { by (9.8) }  \tag{9.17}\\
& =(k-1) \sum_{X \in \tilde{L}_{2}(\mathscr{A})}\binom{\mu(X)+k-2}{k} & & \text { by (9.15). }
\end{align*}
$$

Finally, recall that $\theta_{k}(G(\mathscr{A}))=\phi_{k}(G(\mathscr{A}))$ for $k \leqslant 3$; thus, by Corollary 7.3, equality holds in (9.16) when $k=2$.

## 10. Decomposable Alexander invariants

In this section, we investigate the relationship between the decomposability of an arrangement $\mathscr{A}$, defined in terms of the holonomy Lie algebra $\mathfrak{h}(\mathscr{A})$, and the decomposability of the Alexander-type invariants $B(\mathscr{A}), \widehat{B(\mathscr{A})}$, and $\mathfrak{B}(\mathscr{A})$.
10.1. Decomposable Alexander invariants. Staying with the notations from the previous section, we now make several definitions.

Definition 10.1. We say that the Alexander invariant of a hyperplane arrangement $\mathscr{A}$ is decomposable if the map $\Pi: B(\mathscr{A}) \rightarrow B(\mathscr{A})^{\text {loc }}$ is an isomorphism of $R$-modules.

A similar definition works over the rationals: the Alexander invariant is decomposable over $\mathbb{Q}$ if the map $\Pi \otimes \mathbb{Q}: B(\mathscr{A}) \otimes \mathbb{Q} \rightarrow B(\mathscr{A})^{\mathrm{loc}} \otimes \mathbb{Q}$ is an isomorphism of modules over the ring $R \otimes \mathbb{Q}$. By Theorem 9.1, part (3), this condition is equivalent to $\Pi \otimes \mathbb{Q}$ being injective.

Following [17, §6.4], we say that the completion of the Alexander invariant is decomposable if the map $\widehat{\Pi}: \widehat{B(\mathscr{A})} \rightarrow \widehat{B(\mathscr{A})^{\mathrm{loc}}}$ is an isomorphism of modules over $\hat{R}$. A similar definition works over $\mathbb{Q}$; by Theorem 9.1 , part (2), the $\mathbb{Q}$-decomposability of $\widehat{B(\mathscr{A})}$ is equivalent to the injectivity of $\widehat{\Pi} \otimes \mathbb{Q}$. Clearly, if $B(\mathscr{A})$ is decomposable (over $\mathbb{Z}$ or over $\mathbb{Q}$ ), then $\widehat{B(\mathscr{A})}$ is decomposable (over $\mathbb{Z}$ or over $\mathbb{Q}$ ).

Definition 10.2. We say that the infinitesimal Alexander invariant is decomposable if the map $\bar{\Pi}: \mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}(\mathscr{A})^{\text {loc }}$ is an isomorphism of $S$-modules.

Likewise, the infinitesimal Alexander invariant of $\mathscr{A}$ is decomposable over $\mathbb{Q}$ if the map $\bar{\Pi} \otimes \mathbb{Q}: \mathfrak{B}(\mathscr{A}) \otimes \mathbb{Q} \rightarrow \mathfrak{B}(\mathscr{A})^{\mathrm{loc}} \otimes \mathbb{Q}$ is an isomorphism of modules over $S \otimes \mathbb{Q}$. By Theorem 9.1, part (1), this condition is equivalent to $\bar{\Pi} \otimes \mathbb{Q}$ being injective. Evidently, the property of $\mathfrak{B}(\mathscr{A})$ being decomposable (or $\mathbb{Q}$-decomposable) depends only on the intersection lattice of the arrangement.

These definitions raise the following problem, which is the rough analogue of Question 7.7 in this context.

Question 10.3. For an arrangement $\mathscr{A}$, are the decomposability and $\mathbb{Q}$-decomposability properties of $B(\mathscr{A})$ (and likewise for $\widehat{B(\mathscr{A})}$ and $\mathfrak{B}(\mathscr{A})$ ) equivalent?

There is a connection between decomposability and separability of the Alexander invariant of an arrangement, that we now spell out.

Proposition 10.4. Let $\mathscr{A}$ be an arrangement. If $B(\mathscr{A})$ is decomposable (over $\mathbb{Q}$ ), then $B(\mathscr{A})$ is separated (over $\mathbb{Q}$ ).

Proof. First suppose $B(\mathscr{A})$ is decomposable, that is, $B(\mathscr{A})$ is isomorphic to $B(\mathscr{A})^{\text {loc }}=$ $\oplus_{X \in L_{2}(\mathscr{A})} B\left(\mathscr{A}_{X}\right)_{j_{\sharp}^{X}}$. Since $G\left(\mathscr{A}_{X}\right)=F_{\mu(X)} \times \mathbb{Z}$, it follows from Corollary 2.13 and Lemma 2.14 that $B\left(\mathscr{A}_{X}\right)_{j_{\sharp}^{\chi}}$ is separated. Since completion commutes with direct sums, we conclude that $B(\mathscr{A})$ is separated. The claim over $\mathbb{Q}$ is proved in a similar fashion.
10.2. Alexander invariants of decomposable arrangements. We are now in a position to state and prove the main results of this section. We start with a lemma.

Lemma 10.5. For every arrangement $\mathscr{A}$, there are isomorphisms

$$
\operatorname{gr}_{1}(B(\mathscr{A})) \cong \mathfrak{B}_{1}(\mathscr{A}) \cong \mathfrak{h}_{3}(\mathscr{A})
$$

Proof. Set $G=G(\mathscr{A})$. By Theorem 6.2, Lemma 3.1, and Corollary 3.4, respectively, we have natural isomorphisms

$$
\begin{equation*}
\mathfrak{h}_{3}(G) \xrightarrow{\cong} \operatorname{gr}_{3}(G) \xrightarrow{\cong} \operatorname{gr}_{3}\left(G / G^{\prime \prime}\right) \cong \operatorname{gr}_{1}(B(G)) . \tag{10.1}
\end{equation*}
$$

Moreover, by the grading convention (4.7), we have the equality $\mathfrak{B}_{1}(G)=\mathfrak{h}_{3}(G)$, and this completes the proof.
Theorem 10.6. Let $\mathscr{A}$ be a hyperplane arrangement.
(1) If $B(\mathscr{A})$ is decomposable (over $\mathbb{Q}$ ), then $\mathscr{A}$ is decomposable (over $\mathbb{Q}$ ).
(2) If $\mathfrak{B}(\mathscr{A})$ is decomposable (over $\mathbb{Q}$ ), then $\mathscr{A}$ is decomposable (over $\mathbb{Q}$ ).

Proof. Suppose $B(\mathscr{A})$ is decomposable, that is, the map $\Pi: B(\mathscr{A}) \rightarrow B(\mathscr{A})^{\text {loc }}$ is an isomorphism. Passing to associated graded and taking degree 1 pieces, we obtain an isomorphism, $\operatorname{gr}_{1}(\Pi): \operatorname{gr}_{1}(B(\mathscr{A})) \rightarrow \operatorname{gr}_{1}\left(B(\mathscr{A})^{\text {loc }}\right)$. Using the naturality of the isomorphisms from (10.1), we conclude that the map $\mathfrak{h}_{3}(G) \rightarrow \mathfrak{h}_{3}(G)^{\text {loc }}$ is also an isomorphism, that is, $\mathscr{A}$ is decomposable. Similar arguments work for $\mathfrak{B}(\mathscr{A})$ and over $\mathbb{Q}$.

Note that the rational versions of statements (1) and (2) above are equivalent, since $\operatorname{gr}(B(\mathscr{A})) \otimes \mathbb{Q} \cong \mathfrak{B}(A) \otimes \mathbb{Q}$, but the integral versions of the two statements are a priori different. We now consider the reverse implications of these statements.

Theorem 10.7. Let $\mathscr{A}$ be a hyperplane arrangement.
(1) If $\mathscr{A}$ is decomposable, then $\mathfrak{B}(\mathscr{A})$ is decomposable.
(2) If $\mathscr{A}$ is $\mathbb{Q}$-decomposable, then both $\mathfrak{B}(\mathscr{A})$ and $\widehat{B(\mathscr{A})}$ are $\mathbb{Q}$-decomposable.
(3) If $\mathscr{A}$ is $\mathbb{Q}$-decomposable and $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated, then $B(\mathscr{A})$ is $\mathbb{Q}$-decomposable.

Proof. First suppose $\mathscr{A}$ is decomposable. Let $j: M(\mathscr{A}) \rightarrow \prod_{X \in L_{2}(\mathscr{A})} M\left(\mathscr{A}_{X}\right)$ be the map from (7.1). The homomorphism $j_{\sharp}: G(\mathscr{A}) \rightarrow \prod_{X} G\left(\mathscr{A}_{X}\right)$ induces a map of graded Lie algebras, $\mathfrak{h}\left(j_{\sharp}\right): \mathfrak{h}(\mathscr{A}) \rightarrow \prod_{X} \mathfrak{h}\left(\mathscr{A}_{X}\right)$. Theorem 7.8, part (1) insures that the restriction $\mathfrak{h}^{\prime}\left(j_{\sharp}\right): \mathfrak{h}^{\prime}(\mathscr{A}) \rightarrow \prod_{X} \mathfrak{h}^{\prime}\left(\mathscr{A}_{X}\right)$ is an isomorphism. It follows that the induced map on quotients, $\mathfrak{h}^{\prime}(\mathscr{A}) / \mathfrak{h}^{\prime \prime}(\mathscr{A}) \rightarrow \prod_{X} \mathfrak{h}^{\prime}\left(\mathscr{A}_{X}\right) / \mathfrak{h}^{\prime \prime}\left(\mathscr{A}_{X}\right)$, is an isomorphism of graded abelian groups. As noted previously, this map coincides with the morphism of (graded) $S$-modules $\bar{\Pi}$ from (9.9). Therefore, the map $\bar{\Pi}: \mathfrak{B}(\mathscr{A}) \rightarrow \mathfrak{B}(\mathscr{A})^{\text {loc }}$ is an isomorphism of $S$-modules, and this proves claim (1).

To prove the last two claims, suppose $\mathscr{A}$ is $\mathbb{Q}$-decomposable. Proceeding as above, we infer that the map $\bar{\Pi} \otimes \mathbb{Q}: \mathfrak{B}(\mathscr{A}) \otimes \mathbb{Q} \rightarrow \mathfrak{B}(\mathscr{A})^{\mathrm{loc}} \otimes \mathbb{Q}$ is an isomorphism of $S \otimes \mathbb{Q}$-modules. Therefore, by Lemmas 2.15 and 2.16, its completion, $\widehat{\bar{\Pi}} \otimes \mathbb{Q}$, is also an isomorphism. This map is the bottom arrow from the commuting square (9.13), in which the side arrows are also isomorphisms. Therefore, the top arrow, $\widehat{\Pi} \otimes \mathbb{Q}: \widehat{B(\mathscr{A})} \otimes \mathbb{Q} \rightarrow \widehat{B(\mathscr{A})^{\mathrm{loc}}} \otimes \mathbb{Q}$ is an isomorphism of $\hat{R} \otimes \mathbb{Q}$-modules. If, in addition, the $R \otimes \mathbb{Q}$-module $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated, then, by Lemmas 2.15 and 2.16 again, the map $\Pi \otimes \mathbb{Q}: B(\mathscr{A}) \otimes \mathbb{Q} \rightarrow B(\mathscr{A})^{\text {loc }} \otimes \mathbb{Q}$ is an isomorphism. This completes the proof of claims (2) and (3).

Using different methods, it was shown in [17, Thm. 7.9] that the following implication also holds.
Theorem 10.8 ([17]). If $\mathscr{A}$ is decomposable, then $\widehat{B(\mathscr{A})}$ is decomposable.
Putting things together, we obtain the following corollary.
Corollary 10.9. For an arrangement $\mathscr{A}$, the following are equivalent.
(1) $B(\mathscr{A})$ is decomposable (over $\mathbb{Q}$ ).
(2) $\mathscr{A}$ is decomposable and $B(\mathscr{A})$ is separated (over $\mathbb{Q})$.

Proof. The implication $(1) \Rightarrow(2)$ (over either $\mathbb{Z}$ or $\mathbb{Q}$ ) follows at once from Proposition 10.4 and Theorem 10.6, part (1).

The implication (2) $\Rightarrow$ (1) over $\mathbb{Q}$ is the content of Theorem 10.7, part (3). To prove the same implication over $\mathbb{Z}$, we proceed in a similar manner. To start with, since $\mathscr{A}$ is decomposable, Theorem 10.8 insures that $\widehat{B(\mathscr{A})}$ is decomposable, that is, the map $\widehat{\Pi}: \widehat{B(\mathscr{A})} \rightarrow \widehat{B(\mathscr{A})^{\text {loc }}}$ is an isomorphism of $\widehat{R}$-modules. Now, since $B(\mathscr{A})$ is assumed to be separated, Lemmas 2.15 and 2.16 imply that the map $\Pi: B(\mathscr{A}) \rightarrow B(\mathscr{A})^{\text {loc }}$ is an isomorphism of $R$-modules, and we are done.
10.3. Discussion and examples. The third part of Theorem 10.7 raises the following basic question regarding the Alexander invariants of decomposable arrangements.

Question 10.10. Let $\mathscr{A}$ be an arrangement.
(1) Suppose $\mathscr{A}$ is decomposable. Is then the Alexander invariant $B(\mathscr{A})$ separated in the $I$-adic topology?
(2) If $\mathscr{A}$ is decomposable over $\mathbb{Q}$, is $B(\mathscr{A}) \otimes \mathbb{Q}$ separated?

If the answer to part (2) were to be yes, then we could dispense with the separation hypothesis in Theorem 10.7, part (3), and conclude that $\mathbb{Q}$-decomposability of $\mathscr{A}$ implies $\mathbb{Q}$-decomposability of $B(\mathscr{A})$.

The following result gives a combinatorial criterion for deciding the decomposability of the Alexander invariants of a class of arrangements.

Proposition 10.11. Let $\mathscr{A}$ be an arrangement such that $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}(\mathscr{A}(m))$, for some $r$-tuple $m=\left(m_{1}, \ldots, m_{r}\right)$ with $m_{i} \geqslant 2$. Then $B(\mathscr{A})$ is decomposable.

Proof. By Theorem 8.5, the group $G(\mathscr{A})$ is isomorphic to $G(\mathscr{A}(m))=F_{m_{1}} \times \cdots \times F_{m_{r}} \times \mathbb{Z}$. By Lemma 2.2, $B(\mathscr{A}(m)) \cong B\left(F_{m_{1}}\right)_{p_{1}} \oplus \cdots \oplus B\left(F_{m_{r}}\right)_{p_{r}}$, where $p_{i}$ are the projection maps of $G(\mathscr{A}(m))$ onto its factors. On the other hand, recall that $\widetilde{L}_{2}(\mathscr{A}(m))=\left\{X_{1}, \ldots, X_{r}\right\}$, with $\mu\left(X_{i}\right)=m_{i}$. Consequently, $B\left(F_{m_{i}}\right) \cong B\left(\mathscr{A}_{X_{i}}\right)$, and so $B(\mathscr{A}(m)) \cong B(\mathscr{A}(m))^{\text {loc }}$, showing that $B(\mathscr{A}(m))$ is decomposable. Therefore, $B(\mathscr{A})$ is decomposable.

Remark 10.12. In view of Theorem 10.6, this proposition shows that the arrangements $\mathscr{A}(m)$ are decomposable, thus giving a different proof of Lemma 8.3. Alternatively, one can use Lemma 8.3 together with Theorem 8.5 and Corollary 10.9 to give another proof of Proposition 10.11.
10.4. Chen ranks of $\mathbb{Q}$-decomposable arrangements. As an application of Theorem 10.7, we obtain an explicit formula for the Chen ranks of an arrangement $\mathscr{A}$, provided
that $\mathscr{A}$ is decomposable over $\mathbb{Q}$. Similar formulas were given in [17, Thm. 7.9] and [48, Thm. 6.2] under the (possibly stronger) assumption that $\mathscr{A}$ is decomposable.

Corollary 10.13. Let $\mathscr{A}$ be a $\mathbb{Q}$-decomposable arrangement. Then the Chen ranks of $G(\mathscr{A})$ are given by $\theta_{1}(G(\mathscr{A}))=|\mathscr{A}|$ and

$$
\theta_{k}(G(\mathscr{A}))=\sum_{X \in \tilde{L}_{2}(\mathscr{A})} \theta_{k}\left(G\left(\mathscr{A}_{X}\right)\right)=(k-1) \sum_{X \in \tilde{L}_{2}(\mathscr{A})}\binom{\mu(X)+k-2}{k}
$$

for all $k \geqslant 2$.
Proof. The proof is similar to that of Corollary 9.4. The only difference occurs on the second line of display (9.17), where we need to replace the inequality provided by Theorem 9.1, part (1) with the equality provided by Theorem 10.7, part (2).

## 11. Сономology jump loci and decomposability

In this section, we briefly review the (degree 1) resonance and characteristic varieties of arrangements, and then describe those varieties in a decomposable setting.
11.1. Characteristic varieties. Let $G$ be a finitely generated group. The character group, $\mathbb{T}_{G}=\operatorname{Hom}\left(G, \mathbb{C}^{*}\right)$, is an abelian, complex algebraic group, with identity $\mathbf{1}$ the trivial representation. The coordinate ring of $\mathbb{T}_{G}$ is the group algebra $\mathbb{C}\left[G_{\text {ab }}\right]$; thus, we may identify $\mathbb{T}_{G}$ with $\operatorname{Spec}\left(\mathbb{C}\left[G_{\mathrm{ab}}\right]\right)$, the maximal spectrum of this $\mathbb{C}$-algebra. Since each character $\rho: G \rightarrow \mathbb{C}^{*}$ factors through the abelianization $G_{\mathrm{ab}}$, the map $\mathrm{ab}: G \rightarrow G_{\mathrm{ab}}$ induces an isomorphism, $\mathrm{ab}^{*}: \mathbb{T}_{G_{\mathrm{ab}}} \xrightarrow{\simeq} \mathbb{T}_{G}$.

Let $X$ be a connected CW-complex with finite 1 -skeleton and with $\pi_{1}(X)=G$. Upon identifying a point $\rho \in \mathbb{T}_{G}=H^{1}\left(X ; \mathbb{C}^{*}\right)$ with a rank one local system $\mathbb{C}_{\rho}$ on $X$, we define for each $s \geqslant 1$ the depth $s$ characteristic variety of $X$ as

$$
\begin{equation*}
\mathscr{V}_{s}(X):=\left\{\rho \in H^{1}\left(X ; \mathbb{C}^{*}\right) \mid \operatorname{dim}_{\mathbb{C}} H_{1}\left(X ; \mathbb{C}_{\rho}\right) \geqslant s\right\} . \tag{11.1}
\end{equation*}
$$

Clearly, $\mathbf{1} \in \mathscr{V}_{s}(X)$ if and only if $b_{1}(X) \geqslant s$. Since a classifying space $K(G, 1)$ may be constructed by attaching to $X$ cells of dimension 3 and higher, it is straightforward to verify that $\mathscr{V}_{s}(X)$ coincides with $\mathscr{V}_{s}(G):=\mathscr{V}_{s}(K(G, 1))$, for all $s \geqslant 1$.

Every group homomorphism $\alpha: G \rightarrow H$ induces a morphism between character groups, $\alpha^{*}: \mathbb{T}_{H} \rightarrow \mathbb{T}_{G}$, given by $\alpha^{*}(\rho)(g)=\alpha(\rho(g))$. Now suppose $G$ is finitely generated and $\alpha$ is surjective. Then the morphism $\alpha^{*}$ is injective, and it restricts to an embedding $\mathscr{V}_{s}(H) \hookrightarrow$ $\mathscr{V}_{s}^{1}(G)$, for each $s \geqslant 1$.

It has been known since the work of Libgober [34] and E. Hironaka [30] that the characteristic and Alexander varieties of spaces and groups are intimately related. We record here the result we shall need; a proof valid in all depths $s \geqslant 1$ was given in [61].

Theorem 11.1. Let $G$ be a finitely generated group. Then, for all $s \geqslant 1$,

$$
\begin{equation*}
\mathscr{V}_{s}(G)=\operatorname{supp}\left(\bigwedge^{k} B(G) \otimes \mathbb{C}\right) \tag{11.2}
\end{equation*}
$$

at least away from the identity $\mathbf{1} \in \mathbb{T}_{G}$.
Part of the importance of the characteristic varieties lies in the fact that they control the Betti numbers of regular, finite abelian covers $p: Y \rightarrow X$. For instance, suppose that the deck-transformation group is cyclic of order $N$, classified by an epimorphism $\chi: G=\pi_{1}(X) \rightarrow \mathbb{Z}_{N}$. The induced morphism between character groups, $\chi^{*}: \mathbb{T}_{\mathbb{Z}_{N}} \rightarrow \mathbb{T}_{G}$, is injective, and so $\operatorname{im}\left(\chi^{*}\right) \cong \mathbb{Z}_{N}$. A result proved in various levels of generality in [34, 54, 30, 43, 20] now shows that

$$
\begin{equation*}
b_{1}(Y)=\sum_{s \geqslant 1}\left|\operatorname{im}\left(\chi^{*}\right) \cap \mathscr{V}_{s}(X)\right| . \tag{11.3}
\end{equation*}
$$

11.2. Resonance varieties. Let $G$ be a group, and let $H^{*}=H^{*}(G ; \mathbb{C})$ be its cohomology algebra over $\mathbb{C}$. For our purposes here, we will only consider the truncated algebra $H^{\leqslant 2}$; moreover, we will assume that $b_{1}(G)=\operatorname{dim}_{\mathbb{C}} H^{1}$ is finite. For each element $a \in H^{1}$, we have $a^{2}=0$, and so left-multiplication by $a$ defines a cochain complex,

$$
\begin{equation*}
\left(H, \delta_{a}\right): H^{0} \xrightarrow{\delta_{a}^{0}} H^{1} \xrightarrow{\delta_{a}^{1}} H^{2} \tag{11.4}
\end{equation*}
$$

with differentials $\delta_{a}^{i}(u)=a \cdot u$ for $u \in H^{i}$. The resonance varieties witness the extent to which this complex fails to be exact in the middle. More precisely, for each $s \geqslant 1$, the depth s resonance variety of $G$ is defined as

$$
\begin{equation*}
\mathscr{R}_{s}(G):=\left\{a \in H^{1} \mid \operatorname{dim}_{\mathbb{C}} H^{1}\left(H, \delta_{a}\right) \geqslant s\right\} . \tag{11.5}
\end{equation*}
$$

These sets are homogeneous algebraic subvarieties of the affine space $H^{1}=H^{1}(G ; \mathbb{C})$. Clearly, $\mathbf{0} \in \mathscr{R}_{k}(G)$ if and only if $s \leqslant b_{1}(G)$; in particular, $\mathscr{R}_{1}(G)=\varnothing$ if and only if $b_{1}(G)=0$. Furthermore, we have a descending filtration,

$$
\begin{equation*}
H^{1}(G ; \mathbb{C}) \supseteq \mathscr{R}_{1}(G) \supseteq \mathscr{R}_{2}(G) \supseteq \cdots \supseteq \mathscr{R}_{b}(G) \supseteq \mathscr{R}_{b+1}(G)=\varnothing \tag{11.6}
\end{equation*}
$$

where $b=b_{1}(G)$.
The next result identifies the depth-s resonance variety of a group as the support locus of the $s$-th exterior power of its infinitesimal Alexander invariant.

Theorem 11.2 ([23, 22, 61]). Let $G$ be a group with $b_{1}(G)<\infty$. Then

$$
\begin{equation*}
\mathscr{R}_{s}(G)=\operatorname{supp}\left(\bigwedge^{s} \mathfrak{B}(G) \otimes \mathbb{C}\right) \tag{11.7}
\end{equation*}
$$

for all $s \geqslant 1$, at least away from $\mathbf{0} \in H^{1}(G ; \mathbb{C})$.

Since $\mathfrak{B}(G)=\mathfrak{h}^{\prime}(G) / \mathfrak{h}^{\prime \prime}(G)$, we infer from Theorem 11.2 that the resonance varieties of $G$ only depend on the holonomy Lie algebra $\mathfrak{h}(G)$. More precisely, let $G_{1}$ and $G_{2}$ be two groups with finite first Betti number, and suppose that $\mathfrak{h}\left(G_{1} ; \mathbb{C}\right) \cong \mathfrak{h}\left(G_{2} ; \mathbb{C}\right)$, as graded Lie algebras. There is then a linear isomorphism, $H^{1}\left(G_{1} ; \mathbb{C}\right) \cong H^{1}\left(G_{2} ; \mathbb{C}\right)$, restricting to isomorphisms $\mathscr{R}_{s}\left(G_{1}\right) \cong \mathscr{R}_{s}\left(G_{2}\right)$ for all $s \geqslant 1$.

We shall identify the tangent space to the character group $\mathbb{T}_{G}=H^{1}\left(G ; \mathbb{C}^{*}\right)$ with the linear space $H^{1}(G ; \mathbb{C})$, and we will denote by $\mathrm{TC}_{\mathbf{1}}\left(\mathscr{V}_{s}(G)\right)$ the tangent cone at the identity $\mathbf{1}$ to the characteristic variety $\mathscr{V}_{s}(G)$. It is known that $\mathrm{TC}_{\mathbf{1}}\left(\mathscr{V}_{s}(G)\right)$ is always a (homogeneous) subvariety of the resonance variety $\mathscr{R}_{s}(G)$. The basic relationship between the characteristic and resonance varieties in the 1 -formal setting is encapsulated in the "Tangent Cone formula" from [24, Thm. A], which we recall next.

Theorem 11.3 ([24]). If $G$ is a 1-formal group, then $\mathrm{TC}_{\mathbf{1}}\left(\mathscr{V}_{s}(G)\right)=\mathscr{R}_{s}(G)$, for all $s \geqslant 1$.
11.3. Cohomology jump loci of arrangements. Let $\mathscr{A}$ be a complex hyperplane arrangement. Since complement $M=M(\mathscr{A})$ is a smooth, quasi-projective variety, work of Arapura [1] shows that the characteristic varieties $\mathscr{V}_{s}(M)$ are finite unions of torsiontranslates of algebraic subtori of $H^{1}\left(M ; \mathbb{C}^{*}\right) \cong\left(\mathbb{C}^{*}\right)^{|\mathcal{A}|}$. Furthermore, the projection map $\pi: M \rightarrow \mathbb{P}(M)=U$ induces a monomorphism $H^{1}\left(U ; \mathbb{C}^{*}\right) \hookrightarrow H^{1}\left(M ; \mathbb{C}^{*}\right)$ that restricts to isomorphisms $\mathscr{V}_{s}(U) \xrightarrow{\simeq} \mathscr{V}_{s}(M)$. Since $M$ is also a formal space, each resonance variety $\mathscr{R}_{s}(M)$ coincide with the tangent cone at the trivial character to $\mathscr{V}_{s}(M)$, see [18, 24]. As shown in [28], building on work from [27,35], the varieties $\mathscr{R}_{s}(M)$ may be described solely in terms of multinets on sub-arrangements of $\mathscr{A}$. In general, though, the varieties $\mathscr{V}_{s}(M)$ may contain components which do not pass through the origin, see [57, 11, 20].

Now let $X \in L_{2}(\mathscr{A})$ be a rank-2 flat, and let $\mathscr{A}_{X}$ be localization of $\mathscr{A}$ at $X$. Since $\pi_{1}\left(U\left(\mathscr{A}_{X}\right)\right)=F_{\mu(X)}$, a standard computation shows that, for each $s \geqslant 1$,

$$
\mathscr{V}_{s}\left(U\left(\mathscr{A}_{X}\right)\right)= \begin{cases}H^{1}\left(U\left(\mathscr{A}_{X}\right) ; \mathbb{C}^{*}\right) \cong\left(\mathbb{C}^{*}\right)^{\mu(X)} & \text { if } s<\mu(X)  \tag{11.8}\\ 1 & \text { if } s=\mu(X) \\ \varnothing & \text { if } s>\mu(X)\end{cases}
$$

and likewise for $\mathscr{R}_{s}\left(U\left(\mathscr{A}_{X}\right)\right)=\mathrm{TC}_{\mathbf{1}}\left(\mathscr{V}_{s}\left(U\left(\mathscr{A}_{X}\right)\right)\right)$.
Consider now the inclusion $j_{X}: M(\mathscr{A}) \hookrightarrow M\left(\mathscr{A}_{X}\right)$ and the induced monomorphism of algebraic groups, $\left.\left.j_{X}^{*}: H^{1}\left(M\left(\mathscr{A}_{X}\right)\right) ; \mathbb{C}^{*}\right) \hookrightarrow H^{1}(M(\mathscr{A})) ; \mathbb{C}^{*}\right)$. If $\mu(X)>s$, the map $j_{X}^{*}$ restricts to an isomorphism from $\mathscr{V}_{s}\left(M\left(\mathscr{A}_{X}\right)\right) \cong\left(\mathbb{C}^{*}\right)^{\mu(X)}$ to the subtorus $T_{X} \subset H^{1}\left(M(\mathscr{A}) ; \mathbb{C}^{*}\right)$ given by

$$
\begin{equation*}
T_{X}=\left\{t: \prod_{H_{i} \in \mathscr{A}_{X}} t_{i}=1 \text { and } t_{i}=1 \text { if } H_{i} \notin \mathscr{A}_{X}\right\} . \tag{11.9}
\end{equation*}
$$

Likewise, the map $\left.\left.j_{X}^{*}: H^{1}\left(M\left(\mathscr{A}_{X}\right)\right) ; \mathbb{C}\right) \hookrightarrow H^{1}(M(\mathscr{A})) ; \mathbb{C}\right)$ restricts to an isomorphism from $\mathscr{R}_{s}\left(M\left(\mathscr{A}_{X}\right)\right) \cong \mathbb{C}^{\mu(X)}$ to the linear subspace $L_{X} \subset H^{1}(M(\mathscr{A}) ; \mathbb{C})$ given by

$$
\begin{equation*}
L_{X}=\left\{x: \sum_{H_{i} \in \mathscr{A}_{X}} x_{i}=0 \text { and } x_{i}=0 \text { if } H_{i} \notin \mathscr{A}_{X}\right\} . \tag{11.10}
\end{equation*}
$$

11.4. Cohomology jump loci of decomposable arrangements. The next result gives a complete description of the irreducible components of the resonance and characteristic varieties of $\mathbb{Q}$-decomposable arrangements, under a separation assumption on the rationalized Alexander invariant for the latter varieties.

Theorem 11.4. Let $\mathscr{A}$ be $a \mathbb{Q}$-decomposable arrangement, with complement $M=M(\mathscr{A})$. For each $s \geqslant 1$, the following hold.
(1) $\mathscr{R}_{s}(M)=\underset{\substack{X \in L_{2}(\mathscr{A}) \\ \mu(X)>s}}{ } L_{X}$.
(2) If $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated, then $\mathscr{V}_{s}(M)=\underset{\substack{X \in L_{2}(\mathscr{A}) \\ \mu(X)>s}}{ } T_{X}$.

Proof. By Theorem 11.2, we have that $\mathscr{R}_{s}(M)=\operatorname{supp}\left(\bigwedge^{s} \mathfrak{B}(\mathscr{A}) \otimes \mathbb{C}\right)$. Likewise, for each $X \in L_{2}(\mathscr{A})$, we have that $\mathscr{R}_{s}\left(M\left(\mathscr{A}_{X}\right)\right)=\operatorname{supp}\left(\bigwedge^{s} \mathfrak{B}\left(\mathscr{A}_{X}\right) \otimes \mathbb{C}\right)$, and this variety is either included in $\{\boldsymbol{0}\}$ if $\mu(X) \leqslant s$ or is isomorphic to $L_{X}$ under the embedding $j_{X}^{*}: H^{1}\left(M\left(\mathscr{A}_{X}\right) ; \mathbb{C}\right) \hookrightarrow H^{1}(M ; \mathbb{C})$ if $\mu(X)>s$.

Now, since $\mathscr{A}$ is assumed to be $\mathbb{Q}$-decomposable, it follows from Theorem 10.7, part (2) that $\mathfrak{B}(\mathscr{A}) \otimes \mathbb{C} \cong \mathfrak{B}(\mathscr{A})^{\text {loc }} \otimes \mathbb{C}$, where recall $\mathfrak{B}(\mathscr{A})^{\text {loc }}=\oplus_{X \in L_{2}(\mathscr{A})} \mathfrak{B}\left(\mathscr{A}_{X}\right)_{\bar{j}_{*}^{X}}$. Therefore,

$$
\begin{align*}
\mathscr{R}_{s}(M) & =\operatorname{supp}\left(\bigwedge^{s} \mathfrak{B}(\mathscr{A})^{\mathrm{loc}} \otimes \mathbb{C}\right) \\
& =\bigcup_{X \in L_{2}(\mathscr{A})} j_{X}^{*}\left(\operatorname{supp}\left(\bigwedge^{s} \mathfrak{B}\left(\mathscr{A}_{X}\right) \otimes \mathbb{C}\right)\right)  \tag{11.11}\\
& =\bigcup_{\substack{X \in L_{2}(\mathscr{A}) \\
\mu(X)>s}} L_{X}
\end{align*}
$$

and this proves part (1). The proof of part (2) is entirely similar, using Theorem 11.1 as the initial step and then Theorem 10.7, part (3).

If Question 10.10, part (2) were to have a positive answer, then we could dispense with the separation assumption in part (2) of the above theorem. A priori, though, if $\mathscr{A}$ is $\mathbb{Q}$ decomposable but $B(\mathscr{A}) \otimes \mathbb{Q}$ is not separated, the characteristic variety $\mathscr{V}_{1}(M)$ may have irreducible components that do not pass through the identity of $H^{1}\left(M ; \mathbb{C}^{*}\right)$.
11.5. Chen ranks and resonance. As an application of the above theorem and of Corollary 10.13 , we now show that the Chen Ranks conjecture from [56] holds in the $\mathbb{Q}$-decomposable setting, in the sharpest possible range.

Corollary 11.5. Let $\mathscr{A}$ be a $\mathbb{Q}$-decomposable arrangement. For each $r \geqslant 2$, let $h_{r}$ be the number of irreducible components of $\mathscr{R}_{1}(M(\mathscr{A}))$ of dimension $r$. Then

$$
\theta_{k}(G(\mathscr{A}))=\sum_{r \geqslant 2} h_{r} \theta_{k}\left(F_{r}\right),
$$

for all $k \geqslant 2$.
Proof. It follows from Theorem 11.4, part (1) that the decomposition into irreducible components of $\mathscr{R}_{1}(M(\mathscr{A}))$ is of the form

$$
\begin{equation*}
\mathscr{R}_{1}(M(\mathscr{A}))=\bigcup_{\substack{X \in L_{2}(\mathscr{A}) \\ \mu(X)>1}} L_{X}, \tag{11.12}
\end{equation*}
$$

where each component $L_{X}$ is a $\mathbb{C}$-linear subspace of $H^{1}(M(\mathscr{A}) ; \mathbb{C})$ of dimension $\mu(X)>1$. Hence, $h_{r}=\#\left\{X \in L_{2}(\mathscr{A}): \mu(X)=r\right\}$, for all $r \geqslant 2$.

Now, for each $k \geqslant 2$, we have

$$
\begin{align*}
\theta_{k}(G(\mathscr{A})) & =\sum_{X \in L_{2}(\mathscr{A})} \theta_{k}\left(G\left(\mathscr{A}_{X}\right)\right) \quad \text { by Corollary } 10.13 \\
& =\sum_{r \geqslant 2} \sum_{\substack{X \in L_{2}(\mathscr{A}) \\
\mu(X)=r}} \theta_{k}\left(F_{r}\right) \quad \text { since } G\left(\mathscr{A}_{X}\right) \cong F_{\mu(X)} \times \mathbb{Z}  \tag{11.13}\\
& =\sum_{r \geqslant 2} h_{r} \theta_{k}\left(F_{r}\right),
\end{align*}
$$

and this completes the proof.

## 12. Milnor fibrations of decomposable arrangements

In this final section, we relate the $\mathbb{Q}$-decomposability of an arrangement (under a separability assumption) to the triviality of the algebraic monodromy of its Milnor fibrations.
12.1. The Milnor fibration of a multi-arrangement. Let $\mathscr{A}$ be a central arrangement of hyperplanes in $\mathbb{C}^{d+1}$. To each hyperplane $H \in \mathscr{A}$, we may assign a multiplicity $m_{H} \in$ $\mathbb{N}$. This yields a multi-arrangement $(\mathscr{A}, \mathbf{m})$, where $\mathbf{m}=\left(m_{H}\right)_{H \in \mathscr{A}}$, and a homogeneous polynomial,

$$
\begin{equation*}
f_{\mathbf{m}}=\prod_{H \in \mathscr{A}} f_{H}^{m_{H}} \tag{12.1}
\end{equation*}
$$

of degree $N=\sum_{H \in \mathscr{A}} m_{H}$. The polynomial map $f_{\mathbf{m}}: \mathbb{C}^{d+1} \rightarrow \mathbb{C}$ restricts to a map $f_{\mathbf{m}}: M(\mathscr{A}) \rightarrow$ $\mathbb{C}^{*}$. As shown by Milnor [44] in a much more general context, $f_{\mathbf{m}}$ is the projection map
of a smooth, locally trivial bundle, known as the (global) Milnor fibration of the multiarrangement $(\mathscr{A}, \mathbf{m})$,

$$
\begin{equation*}
F_{\mathbf{m}} \longrightarrow M \xrightarrow{f_{\mathbf{m}}} \mathbb{C}^{*} . \tag{12.2}
\end{equation*}
$$

The typical fiber of this fibration, $f_{\mathbf{m}}^{-1}(1)$, is a smooth manifold of dimension $2 d$, called the Milnor fiber of the multi-arrangement, denoted $F_{\mathbf{m}}=F_{\mathbf{m}}(\mathscr{A})$. It is readily seen that $F_{\mathbf{m}}$ is a Stein domain of complex dimension $d$, and thus has the homotopy type of a finite CWcomplex of dimension at most $d$. Moreover, $F_{\mathbf{m}}$ is connected if and only if $\operatorname{gcd}(\mathbf{m})=1$, a condition we will assume henceforth. In the case when all the multiplicities $m_{H}$ are equal to 1 , the polynomial $f=f_{\mathbf{m}}$ is the usual defining polynomial, of degree $n=|\mathscr{A}|$, and $F=F_{\mathrm{m}}$ is the usual Milnor fiber of $\mathscr{A}$.
12.2. Milnor fibers as finite cyclic covers. The monodromy of the Milnor fibration is the self-diffeomorphism $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ given by $z \mapsto e^{2 \pi \mathrm{i} / N} z$. Clearly, the complement $M$ is homotopy equivalent to the mapping torus of $h$. The map $h$ generates a cyclic group of order $N=\sum_{H \in \mathscr{A}} m_{H}$ which acts freely on $F_{\mathbf{m}}$. The quotient space, $F_{\mathbf{m}} / \mathbb{Z}_{N}$, may be identified with the projective complement, $U=\mathbb{P}(M)$, in a way so that the projection map, $\sigma_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}} / \mathbb{Z}_{N}$, coincides with the restriction of the projectivization map, $\pi: M \rightarrow$ $U=\mathbb{P}(M)$, to the subspace $F_{\mathbf{m}}$.

The induced homomorphism on fundamental groups, $\left(f_{\mathbf{m}}\right)_{\sharp}: \pi_{1}(M) \rightarrow \pi_{1}\left(\mathbb{C}^{*}\right)$ may be identified with the map $\mu_{\mathrm{m}}: \pi_{1}(M) \rightarrow \mathbb{Z}, \gamma_{H} \mapsto m_{H}$, which descends to an epimorphism,

$$
\begin{equation*}
\chi_{\mathbf{m}}: \pi_{1}(U) \longrightarrow \mathbb{Z}_{N}, \quad \bar{\gamma}_{H} \mapsto m_{H} \bmod N . \tag{12.3}
\end{equation*}
$$

As shown in $[14,58,60]$, the regular, $N$-fold cyclic cover $\sigma_{\mathbf{m}}: F_{\mathbf{m}} \rightarrow U$ is classified by this epimorphism. In particular, the usual Milnor fiber $F=F(\mathscr{A})$ is classified by the "diagonal" homomorphism, $\chi: \pi_{1}(U) \rightarrow \mathbb{Z}_{n}$, given by $\chi\left(\bar{\gamma}_{H}\right)=1$, for all $H \in \mathscr{A}$.
12.3. Trivial algebraic monodromy. Much effort has been put into computing the homology groups of the Milnor fiber $F_{\mathrm{m}}$ and finding the eigenvalues of the algebraic monodromy $h_{*}$ acting on $H_{*}\left(F_{\mathbf{m}} ; \mathbb{C}\right)$; see for instance $[2,10,14,20,49,60,68]$ and the references therein. Despite some progress, the problem of computing even $b_{1}(F(\mathscr{A}))$ remains open for arbitrary arrangements $\mathscr{A}$.

In recent work [62], we studied in depth those hyperplane arrangements for which the monodromy of the Milnor fibration acts trivially on the first homology of the Milnor fiber (either with $\mathbb{Z}$ or with $\mathbb{Q}$ coefficients). The description of $F_{\mathbf{m}}$ as a finite cyclic cover of $U=U(\mathscr{A})$ makes it apparent that the map $h_{*}: H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right) \rightarrow H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right)$ is the identity if and only if $b_{1}\left(F_{\mathbf{m}}\right)=b_{1}(U)$.

The main result of this section is the next theorem, which gives a sufficient condition for the monodromy in degree 1 to be trivial over $\mathbb{Q}$. The result follows from Theorem 11.4 and [62, Prop. 4.5]; for completeness, we provide a self-contained proof.

Theorem 12.1. Let $\mathscr{A}$ be an arrangement of rank 3 or higher. Suppose $\mathscr{A}$ is $\mathbb{Q}$-decomposable and $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated. Then, for any choice of multiplicities $\mathbf{m}$ on $\mathscr{A}$, the algebraic monodromy of the Milnor fibration, $h_{*}: H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right) \rightarrow H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right)$, is trivial.

Proof. Fix an ordering $H_{1}, \ldots, H_{n}$ of the hyperplanes in $\mathscr{A}$. Then $H^{1}\left(M ; \mathbb{C}^{*}\right)$ may be identified with $\left(\mathbb{C}^{*}\right)^{n}$, with coordinates $t=\left(t_{1}, \ldots, t_{n}\right)$ and $H^{1}\left(U ; \mathbb{C}^{*}\right)$ may be identified with $\left(\mathbb{C}^{*}\right)^{n-1}$, with coordinates $\left(t_{1}, \ldots, t_{n-1}\right)$. The morphism $\pi^{*}: H^{1}\left(U ; \mathbb{C}^{*}\right) \hookrightarrow H^{1}\left(M ; \mathbb{C}^{*}\right)$ may then be viewed as the monomial map $\left(\mathbb{C}^{*}\right)^{n-1} \hookrightarrow\left(\mathbb{C}^{*}\right)^{n}$ which sends $\left(t_{1}, \ldots, t_{n-1}\right)$ to $\left(t_{1}, \ldots, t_{n-1}, t_{1}^{-1} \cdots t_{n-1}^{-1}\right)$.

By formula (11.1), we have that $b_{1}\left(F_{\mathbf{m}}\right)=\sum_{s \geqslant 1}\left|\operatorname{im}\left(\chi_{\mathbf{m}}^{*}\right) \cap \mathscr{V}_{s}(U)\right|$, where $\chi_{\mathbf{m}}: \pi_{1}(U) \rightarrow$ $\mathbb{Z}_{N}$ is the homomorphism from (12.3). The cyclic subgroup $\operatorname{im}\left(\chi_{\mathbf{m}}^{*}\right) \subset\left(\mathbb{C}^{*}\right)^{n-1}$ is generated by the character $\rho=\left(\zeta^{m_{1}}, \ldots, \zeta^{m_{n-1}}\right)$, where $\zeta=e^{2 \pi \mathrm{i} / N}$ and $m_{i}=m_{H_{i}}$. It follows that $\pi^{*}\left(\operatorname{im}\left(\chi_{\mathbf{m}}^{*}\right)\right)$ is the cyclic subgroup of $\left(\mathbb{C}^{*}\right)^{n}$ generated by $\tilde{\rho}=\left(\zeta^{m_{1}}, \ldots, \zeta^{m_{n-1}}, \zeta^{m_{n}}\right)$, and thus is contained in the subtorus $T_{\mathbf{m}}=\left\{\left(z^{m_{1}}, \ldots, z^{m_{n}}\right) \mid z \in \mathbb{C}^{*}\right\} \subset\left(\mathbb{C}^{*}\right)^{n}$.

Now let $C$ be an irreducible component of $\mathscr{V}_{1}(M)$. By assumption, $\mathscr{A}$ is $\mathbb{Q}$-decomposable and $B(\mathscr{A}) \otimes \mathbb{Q}$ is separated; therefore, Theorem 11.4, part (2) insures that $C=T_{X}$, for some 2-flat $X \in L_{2}(\mathscr{A})$. Moreover, since $\mathscr{A}$ is assumed to be of rank at least 3, we have that $\mathscr{A}_{X}$ is properly contained in $\mathscr{A}$, and thus $C$ lies in a proper coordinate subtorus of $H^{1}\left(M ; \mathbb{C}^{*}\right)=\left(\mathbb{C}^{*}\right)^{n}$; hence, $C$ intersects intersects $T_{\mathrm{m}}$ only at the identity. It follows that $\pi^{*}\left(\operatorname{im}\left(\chi_{\mathbf{m}}^{*}\right)\right) \cap \mathscr{V}_{1}(M)=\{\mathbf{1}\}$, and therefore $\operatorname{im}\left(\chi_{\mathbf{m}}^{*}\right) \cap \mathscr{V}_{1}(U)=\{\mathbf{1}\}$. It follows that $b_{1}\left(F_{\mathbf{m}}\right)=$ $n-1$, which is equivalent to the monodromy action on $H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right)$ being trivial.

As an application of this theorem, we give a quick proof of a result of Venturelli, who showed in [67, Thm. 3] that the usual Milnor fibration of a certain class of arrangements has trivial algebraic monodromy (our proof works as well for all the fibers $F_{\mathbf{m}}$ ). Let $\mathscr{A}$ be a central arrangement in $\mathbb{C}^{3}$, and let $\overline{\mathscr{A}}=\mathbb{P}(\mathscr{A})$ be the projectivized arrangement of lines in $\mathbb{C P}^{2}$.
Proposition 12.2 ([67]). Suppose $\overline{\mathscr{A}}$ has two multiple points, $P_{1}$ and $P_{2}$, such that every line in $\overline{\mathscr{A}}$ passes through either $P_{1}$ or $P_{2}$. Then $h_{*}: H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right) \rightarrow H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right)$ is the identity.

Proof. Let $m_{i} \geqslant 3$ be the multiplicity of $P_{i}$. The hypothesis can be rephrased as saying that $L_{\leqslant 2}(\mathscr{A}) \cong L_{\leqslant 2}\left(\mathscr{A}\left(m_{1}, m_{2}\right)\right)$. By Proposition 10.11 , the Alexander invariant $B(\mathscr{A})$ is decomposable and separated. The claim now follows from Theorem 12.1.
12.4. Discussion. It is natural to ask whether one can work over the integers in Theorem 12.1, and also whether one can dispense with the separation hypothesis on the Alexander invariant (either over $\mathbb{Z}$ or over $\mathbb{Q}$ ). More precisely, we have the following question.

Question 12.3. Let $(\mathscr{A}, \mathbf{m})$ be an multi-arrangement, and let $h: F_{\mathbf{m}} \rightarrow F_{\mathbf{m}}$ be the monodromy of the corresponding Milnor fibration.
(1) If $\mathscr{A}$ is decomposable, is the monodromy action on $H_{1}\left(F_{\mathbf{m}} ; \mathbb{Z}\right)$ trivial?
(2) If $\mathscr{A}$ is decomposable over $\mathbb{Q}$, is the monodromy action on $H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right)$ trivial?

If Question 10.10, part (2) were to have a positive answer, then, by Theorem 12.1, the answer to part (2) of the above question would be yes. In general, the group $H_{1}\left(F_{\mathbf{m}} ; \mathbb{Z}\right)$ may have torsion (see [10,20]), even for the usual Milnor fiber $F=F(\mathscr{A})$ when $m_{H}=1$ for all $H \in \mathscr{A}$ (see [68]); thus, the monodromy $h$ may act trivially on $H_{1}\left(F_{\mathbf{m}} ; \mathbb{Q}\right)$ but not on $H_{1}\left(F_{\mathbf{m}} ; \mathbb{Z}\right)$. Nevertheless, we do not know whether this can happen within the class of (Q-)decomposable arrangements, though we suspect that it cannot.

We conclude with an example worth pondering.
Example 12.4. Let $\mathscr{A}$ be the non-Pappus arrangement of 9 hyperplanes in $\mathbb{C}^{3}$ realizing the $\left(9_{3}\right)_{2}$ configuration described in Section 8.4. This arrangement is decomposable, yet we do not know whether the Alexander invariant $B(\mathscr{A})$ is separated, even over $\mathbb{Q}$. Nevertheless, as first shown in [2] and later by different methods in [14], the monodromy of $\mathscr{A}$ acts trivially on $H_{1}(F ; \mathbb{Q})$, that is, $b_{1}(F)=8$. In fact, as noted in [57, Example 10.10], all the components of the characteristic variety $\mathscr{V}_{1}(M(\mathscr{A}))$ are of the form $T_{X}$, for some $X \in$ $\widetilde{L}_{2}(\mathscr{A})$; hence, by the argument in the proof of Theorem 12.1, this also gives $b_{1}(F)=8$. On the other hand, the closely related Pappus arrangement, $\mathscr{A}^{\prime}$ (a realization the $\left(9_{3}\right)_{1}$ configuration), has intersection lattice with the same Möbius function as $\mathscr{A}$, yet $L\left(\mathscr{A}^{\prime}\right) \not \equiv$ $L(\mathscr{A})$. Moreover, $\mathscr{A}^{\prime}$ is not $\mathbb{Q}$-decomposable, and, as shown in [2, 14], $b_{1}\left(F^{\prime}\right)=10$.

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[^0]:    Date: April 9, 2024.
    2010 Mathematics Subject Classification. Primary 52C35. Secondary 16W70, 17B70, 20F14, 20F40, 32S55, 57M07.

    Key words and phrases. Hyperplane arrangement, decomposable arrangement, intersection lattice, fundamental group, holonomy Lie algebra, associated graded Lie algebra, Chen ranks, Alexander invariant, cohomology jump loci, Milnor fibration.
    ${ }^{1}$ Supported in part by Simons Foundation Collaboration Grants for Mathematicians \#693825.

