Be sure to *fully justify your response* to each problem by citing any results in the text that you use and by writing out additional arguments as needed.

1. (10 pts) Determine the values of a for which $\lim_{n\to+\infty}\left(1+\frac{a}{n}\right)^n$ is finite, and give a formula for the limit for those values of a.

   Solution: Note that for $n > |a|$ we have $1 + a/n > 0$. Then $\ln(1 + a/n)$, and we can consider the sequence $\ln[(1 + a/n)^n] = n \ln[1 + a/n]$. The first step is to find the limit $\lim_{n\to\infty} n \ln[1 + a/n]$.

$$\begin{aligned}\lim_{n\to\infty} n\ln[1+a/n] &= \lim_{n\to\infty}\frac{\ln[1+a/n]}{1/n}\\ &= \lim_{n\to\infty}\frac{\left(\frac{1}{1+a/n}\right)(-a/n^2)}{-1/n^2}\\ &= \lim_{n\to\infty} a\left(\frac{1}{1+a/n}\right)\\ &= a\left(\frac{1}{1+0}\right)\\ &= a\end{aligned}$$

   where the second line follows from the first line by using L'Hosptial's rule.

   Since e^x is a continuous function and the sequence $n \ln[1 + a/n]$ converges to a, we have that

(1) $$\lim_{n\to\infty} e^{n\ln[1+a/n]} = e^a$$

   The result now follows from equation (1) and the property that $e^{\ln(y)} = y$ for all $y \in \mathbb{R}$ from which it follows that

$$e^{n\ln[1+a/n]} = e^{\ln(1+a/n)^n} = (1+a/n)^n$$

2. (15 pts) Let f be a function defined on $\mathbb{R}$. Suppose there exists $p > 1$ with the property that $|f(x) - f(y)| \leq |x - y|^p$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

   Solution: Let $x_0 \in \mathbb{R}$, then

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq |x-x_0|^{p-1} \quad \text{for all } x \in R, x \neq x_0$$

   Since $p - 1 > 0$, we have $\lim_{x\to x_0} |x - x_0|^{p-1} = 0$. From the squeeze lemma applied to the inequality

$$0 \leq \left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq |x-x_0|^{p-1}$$

   it now follows that

$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = 0 \quad \text{for all } x_0 \in \mathbb{R}$$

and the result follows since a consequence of the Mean Value Theorem is that if the derivative of function is 0 for all x in an open interval, then the function is constant on that open interval.

3. (10 pts) Let f be a function that is differentiable on an open interval (a, b). Show that if there is a number $M > 0$ such that

$$|f'(x)| \leq M \quad \text{for all } x \in (a, b)$$

then f is uniformly continuous on (a, b).

Solution: From the definition of uniform continuity, it suffices to show that given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \text{for all } x, y \in (a, b) \text{ with } |x - y| < \delta$$

This can be done as follows. Let $\epsilon > 0$ be given and choose $\delta = \epsilon/M$. Let x and y be elements in (a, b) with $x \neq y$. Then by the Mean Value Theorem

$$\frac{f(x) - f(y)}{x - y} = f'(c) \quad \text{for some } c \text{ between } x \text{ and } y$$

then since $|f'(c) \leq M$, it follows that for $|x - y| < \delta = \epsilon/M$ we have

$$|f(x) - f(y)| \leq M|x - y| < M\left(\frac{\epsilon}{M}\right) = \epsilon$$

and the proof is complete.

4. (15 pts) Suppose f is a continuous function on $[a, b]$ and differentiable on the interior (a, b) with constant derivative $f'(x) = M$. Prove using the Mean Value Theorem that $f(x)$ is a linear function (i.e., there are constants A, B such that $f(x) = Ax + B$).

Solution: Let $L(x) = f(a) + M(x - a) = f(a) - Ma + Mx$, then $L(x)$ has the form $Ax + B$ with $A = f(a) - Ma$ and $B = M$. Thus, it suffices to show that $L(x) = f(x)$ for all $x \in [a, b]$.

Set $h(x) = L(x) - f(x)$. Note that h is continuous on $[a, b]$, and differentiable on (a, b) with

$$h'(c) = L'(c) - f'(c) = M - M = 0 \quad \text{for all } c \in (a, b). \tag{2}$$

Moreover, $h(a) = L(a) - f(a) = f(a) + M(a - a) - f(a) = 0$.

Now let $x \in (a, b]$ then by the Mean Value theorem

$$\frac{h(x) - h(a)}{x - a} = h'(c) \quad \text{for some } c \in (a, x)$$

From equation (2) we have $h'(c) = 0$, so $h(x) - h(a) = 0$ for all $x \in (a, b]$. Thus, $h(x) = h(a)$ for all $x \in (a, b]$, and we have

$$h(x) = h(a) = L(a) - f(a) = 0 \quad \text{for all } x \in (a, b]$$

Since $h(x) = L(x) - f(x)$, we have that $L(x) = f(x)$ for all $x \in (a, b]$. Moreover, since $L(a) = f(a)$, it follows that $L(x) = f(x)$ for all $x \in [a, b]$, and the proof is complete.

5. Let $f(x) = \ln(1+x)$, let $\sum_{n=0}^{\infty} a_n x^n$ be the Taylor series for f, and let $R_n(x)$ be the remainder $R_n(x) = \ln(1+x) - \sum_{k=0}^{n-1} a_k x^k$ for $x > -1$.

(a) (10 pts) Using the formula for $R_n(x)$ in §31.3 Taylor's Theorem (p. 250), find an upper bound for $|R_n(x)|$.

Solution: By §31.3 Taylor's Theorem we have

$$\left| \ln(1+x) - \sum_{k=0}^{n-1} a_k x^k \right| = |R_n(x)| = \left| \left(\frac{f^{(n)}(y)}{n!} \right) x^n \right|$$

for some y between 0 and x. Note that $f^{(1)}(x) = (1+x)^{-1}$. If we assume that $f^{(n)}(x) = (-1)^{n-1}[(n-1)!](1+x)^{-n}$, then

$$\begin{aligned} f^{(n+1)}(x) &= (-1)^{n-1}[(n-1)!](-n)(1+x)^{-(n+1)} \\ &= (-1)^n [n!](1+x)^{-(n+1)} \end{aligned}$$

So it follows by mathematical induction that $f^{(n)}(x) = (-1)^{n-1}[(n-1)!](1+x)^{-n}$, and hence,

$$\begin{aligned} |R_n(x)| &= \left| \left(\frac{f^{(n)}(y)}{n!} \right) x^n \right| \\ &= \left| \frac{(n-1)!}{(n!)(1+y)^n} x^n \right| \\ &= \left| \left(\frac{1}{n} \right) \left(\frac{x}{1+y} \right)^n \right| \end{aligned}$$

for some y between 0 and x.

Now note that $\ln(1+x)$ is defined only for $x > -1$ so $1+y > 0$ and since the derivative with respect to y of $(1+y)^{-n}$ is $(-n)(1+y)^{-(n+1)}$ we then have that $(1+y)^{-n}$ is a decreasing function of y, and hence its maximum value over any closed interval is its value at the left hand end point of the interval.

If $x > 0$, then the interval is $[0, x]$, so the left hand end point is 0. In this case, we have $(1+y)^{-n} \leq (1+0)^{-n} = 1$, and hence

$$|R_n(x)| \leq \frac{x^n}{n} \quad \text{for } x > 0 \tag{3}$$

If $-1 < x < 0$ then the interval is $[x, 0]$, the left hand end point is x. In this case, we have $(1+y)^{-n} \leq (1+x)^{-n}$, and hence

$$|R_n(x)| \leq \left(\frac{1}{n} \right) \left| \frac{x}{1+x} \right|^n \quad \text{for } -1 < x < 0 \tag{4}$$

The upper bound for $|R_n(x)|$ is given by the inequality in equation (3) in the case $x > 0$ and by the inequality in equation (4) in the case $-1 < x < 0$.

(b) (10 pts) Find all values of $x > -1$ for which it follows from your result in part (a) that $\lim_{n\to+\infty} R_n(x) = 0$.

Solution: As a first step, note that

(5) $$\lim_{n\to\infty}\left|\frac{r^n}{n}\right| = 0 \quad \text{if and only if} \quad |r| \leq 1$$

To prove the result in equation (5) note that if $|r| \leq 1$, then $0 \leq |r|^n/n \leq 1/n$ and the result follows from the squeeze lemma. If $|r| > 1$, then L'Hospital's rule applies. $d(|r|^n)/dn = \ln|r| \cdot r^n$. Thus, it follows from L'Hospital's rule that $\lim_{n\to\infty} |r|^n/n = +\infty$, and the proof of the result in equation (5) is complete.

From equations (3) and (5) it follows that for $x > 0$, we have $\lim_{n\to\infty} R_n(x) = 0$ if and only if $0 < x \leq 1$.

From equations (4) and (5) it follows that for $-1 < x < 0$, we have $\lim_{n\to infty} R_n(x) = 0$ if and only if $|x/(1+x)| \leq 1$. Note that for $-1 < x < 0$ we have $|x| = -x$ and $1 + x > 0$. Thus

$$\left|\frac{x}{1+x}\right| = \frac{-x}{1+x}$$

so the condition that $|x/(1+x)| \leq 1$ is the same as

$$\begin{aligned} \frac{-x}{1+x} &\leq 1 \\ -x &\leq 1+x \\ 0 &\leq 1+2x \\ -1 &\leq 2x \\ \frac{-1}{2} &\leq x \end{aligned}$$

Thus, the values of x for which it follows that from the result in part (a) that $\lim_{n\to\infty} R_n(x) = 0$ is the closed interval $[-1/2, 1]$.

6. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{2\pi x}\right), & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(a) (10 pts) Is f continuous?

Solution: Answer: Yes. We will show that f is continuous at every $x_0 \in \mathbb{R}$, that is,

(6) $$\lim_{x\to x_0} f(x) = x_0.$$

If $x_0 \neq 0$, then clearly $\lim_{x\to x_0} x = x_0$. Moreover, $\lim_{x\to x_0} \sin\left(\frac{1}{2\pi x}\right) = \sin\left(\frac{1}{2\pi x_0}\right)$, since both the sine function and the function $x \mapsto \frac{1}{2\pi x}$ are continuous. Since products of continuous functions are continuous, (6) holds when $x_0 \neq 0$.

It remains to show that (6) holds when $x_0 = 0$. First note the following. For all $x \neq 0$, we have

$$|f(x)| = \left| x \sin\left(\frac{1}{2\pi x}\right) \right| = |x| \left| \sin\left(\frac{1}{2\pi x}\right) \right| \leq |x|.$$

Moreover, $|f(0)| = 0 \leq 0$. Thus, $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$. Therefore,

$$0 \leq \lim_{x\to 0} |f(x)| \leq \lim_{x \to 0} |x| = 0,$$

and so $\lim_{x\to 0} |f(x)| = 0$ by the Squeeze Theorem for limits of sequences. This implies $\lim_{x\to 0} f(x) = 0$, and this completes the proof of the claim.

(b) (10 pts) Is the restriction of f to the interval $[-1, 1]$ uniformly continuous?

Solution: Since the map f is continuous for all $x \in \mathbb{R}$, it is also the case that f is continuous for all $-1 \leq x \leq 1$; that is, the restriction of f to the interval $[-1, 1]$ is also continuous. Furthermore, note that the interval $[-1, 1]$ is closed.

Now, every continuous function defined on a closed interval is uniformly continuous (Theorem 19.2 in the book). Therefore, restriction of f to the interval $[-1, 1]$ uniformly continuous.

(c) (10 pts) Is f differentiable?

Solution: Answer: No. More precisely, the function f is differentiable at any point $x_0 \neq 0$, since the sine function is differentiable, and the fact that differentiability is preserved under composition and multiplication. But f is *not* differentiable at $x_0 = 0$. Indeed, the Newton quotient at 0 is equal to

$$\frac{f(x) - f(0)}{x - 0} = \frac{x \sin\left(\frac{1}{2\pi x}\right)}{x} = \sin\left(\frac{1}{2\pi x}\right),$$

and

$$\lim_{x\to 0} \sin\left(\frac{1}{2\pi x}\right)$$

does not exist. To see why that is the case, take for instance the sequence $x_n = \frac{1}{4\pi^2 n}$, which converges to 0. Then

$$\lim_{n\to\infty} \sin\left(\frac{1}{2\pi \frac{1}{4\pi^2 n}}\right) = \lim_{n\to\infty} \sin(2\pi n) = \lim_{n\to\infty} 0 = 0.$$

On the other hand, if we take the sequence $x_n = \frac{1}{\pi^2(4n+1)}$, which also converges to 0, then

$$\begin{aligned} \lim_{n\to\infty} \sin\left(\frac{1}{2\pi \frac{1}{\pi^2(4n+1)}}\right) &= \lim_{n\to\infty} \sin\left(\frac{\pi(4n+1)}{2}\right) \\ &= \lim_{n\to\infty} \sin\left(2n\pi + \frac{\pi}{2}\right) \\ &= \lim_{n\to\infty} 1 = 1. \end{aligned}$$

This completes the proof that $\lim_{x\to 0} \sin\left(\frac{1}{2\pi x}\right)$ does not exist, and thus, that the function f is not differentiable at 0.