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MATH 3175 **Group Theory** **Spring 2024**

Solutions to Homework 6

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1. Let G be a group acting on a set S, and let $\phi\colon G \to \mathrm{Sym}(S)$ be the homomorphism defined by $\phi(a) = \lambda_a$ for $a \in G$, where $\lambda_a\colon S \to S$ is the bijection given by $\lambda_a(x) = a * x$ for $x \in S$. For each $x \in S$, let $G_x = \{a \in G : a * x = x\}$ be the stabilizer subgroup of x. Show that the kernel of ϕ coincides with the intersection of all the stabilizer subgroups; that is,

$$\ker(\phi) = \bigcap_{x\in S} G_x\,.$$

First note that

$$\ker(\phi) = \{a \in G : \lambda_a = \mathrm{id}_S\} = \{a \in G : a * x = x,\ \forall x \in S\}.$$

On the other hand,

$$\bigcap_{x\in S} G_x = \bigcap_{x\in S}\{a \in G : a * x = x\} = \{a \in G : a * x = x,\ \forall x \in S\}.$$

Thus, the two sets coincide.

2. Let G be a group, and let $H \le G$ be a non-trivial subgroup. Consider the action of the group H on the set G given by left-multiplication; that is, $h * g = hg$ for $h \in H$ and $g \in G$.

(i) What are the orbits of this action?

For each $g \in G$, the (left) H-orbit of g is $Hg = \{hg : h \in H\}$ — a *right* coset of H. Thus, the orbits of this action are the right cosets of H.

(ii) What are the stabilizer subgroups of this action?

The stabilizer of an element $g \in G$ is the subgroup of H given by $H_g = \{h \in H : hg = g\}$. Since $hg = g$ is equivalent to $h = e$, we conclude that $H_g = \{e\}$. Thus, all the stabilizer subgroups of this action are equal to the trivial subgroup.

(iii) What is the subset of G left fixed by this action?

The fixed subset of this action is $G^H = \{g \in G : hg = g, \forall h \in H\}$. Since by assumption H is not the trivial group, there must be an element $h \in H$ with $h \neq e$, and for this element, the equality $hg = g$ does not hold. Therefore, $G^H = \emptyset$; that is, the action has no fixed points.

3. Let $G = \mathrm{GL}_2(\mathbb{Z}_2)$ be the (multiplicative) group of invertible 2×2 matrices with entries in $\mathbb{Z}_2$. Let $S = \mathbb{Z}_2 \times \mathbb{Z}_2$, viewed as the set of 2×1 vectors with entries in $\mathbb{Z}_2$. Consider the action of G on S given by matrix multiplication on vectors; that is, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \quad \text{and} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in S,$$

then

$$A * \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix}.$$

(i) For each element $\vec{v} \in S$, determine the orbit $G\vec{v}$ and stabilizer $G_{\vec{v}}$.

First note that

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

which is a group of order 6 (in fact, $G \cong S_3$).

If $\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $G\vec{v} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ and $G_{\vec{v}} = G$.

If $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $G\vec{v} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ and $G_{\vec{v}} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$.

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Note that, in each case, the Orbit–Stabilizer Theorem is satisfied, that is, $|G\vec{v}| = |G| / |G_{\vec{v}}|$. Indeed, in the first case, $1 = 6/6$, while in the other 3 cases, $3 = 6/2$.

(ii) Determine the set S^G (the subset of S fixed by G).

$S^G = \{\vec{v} \in S : A\vec{v} = \vec{v},\ \forall A \in G\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

4. Let the symmetric group $G = S_5$ act on itself by conjugation. Consider the permutation $\sigma = (123)(45) \in S_5$.

(i) What is the size of the orbit of σ?

The orbit of σ under the conjugation action is $S_5 \cdot \sigma = \{g\sigma g^{-1} : g \in S_5\}$. This is the conjugacy class of σ in S_5, denoted by $\mathrm{Cl}(\sigma)$. Such a conjugacy class consists of all permutations of the set $\{1, 2, 3, 4, 5\}$ that have the same cycle decomposition as σ, that is, are of type $(\cdots)(\cdot\cdot)$. There are $\binom{5}{3} \cdot 2 = 20$ such permutations:

$$\begin{aligned} \mathrm{Cl}(\sigma) = \{&(123)(45), (132)(45), (124)(35), (142)(35), \\ &(125)(34), (152)(34), (134)(25), (143)(25), \\ &(135)(24), (153)(24), (145)(23), (154)(23), \\ &(234)(15), (243)(15), (235)(14), (253)(14), \\ &(245)(13), (254)(13), (345)(12), (354)(12)\}. \end{aligned}$$

(ii) What are the orders of the elements in this orbit?

All elements in $\mathrm{Cl}(\sigma)$ have the same order as σ, namely, $\mathrm{lcm}(3, 2) = 3 \cdot 2 = 6$.

(iii) What is the size of the stabilizer of σ?

By the Orbit–Stabilizer Theorem,

$$|(S_5)_\sigma| = \frac{|S_5|}{|S_5 \cdot \sigma|} = \frac{5!}{|\mathrm{Cl}(\sigma)|} = \frac{120}{20} = 6.$$

5. Let G be a group acting on a set S.

(i) Suppose G has size 21 and S has size 8. Show that $S^G \neq \emptyset$.

Suppose $S^G = \emptyset$. By the Class Equation for the G-action on S, we have:

$$|S| = \left|S^G\right| + \sum_{|Gx|>1} [G : G_x]$$

Since $|S| = 8$ and $\left|S^G\right| = |\emptyset| = 0$ by assumption, we get

$$\sum_{|Gx|>1} [G : G_x] = 8.$$

On the other hand, from Lagrange's Theorem we know that the index $[G : G_x]$ divides $|G| = 21$, and so $[G : G_x] \in \{1, 3, 7, 21\}$. Since the above sume is equal to $8 < 21$, this rules out $[G : G_x] = 21$. Moreover, by the Orbit–Stabilizer Theorem $[G : G_x] = |G| \,/\, |G_x| = |Gx| > 1$, so this rules out $[G : G_x] = 21$. We are left with only two possibilities: $[G : G_x] \in \{3, 7\}$. Therefore, we need to solve the equation

$$3m + 7n = 8$$

for positive integers m, n. But clearly this equation has no such solution. Therefore, the assumption that $S^G = \emptyset$ has led to a contradiction. This shows that $S^G \neq \emptyset$.

(ii) Give an example where $|G| = 8$, $|S| = 8$, and $S^G = \emptyset$.

Let G be any group of order 8 (e.g., $G = \mathbb{Z}_8$, or D_8, or Q_8, etc), and let $S = G$, with G acting on itself by left-multiplication, that is, $g * x = gx$. Then $S^G = \{x \in G : gx = x,\ \forall g \in G\} = \emptyset$.