**FINAL EXAM**
**MATH 3175 GROUP THEORY**
**SPRING 2024**

(1) Suppose $G = \langle g \rangle$ is a cyclic group of order 30 generated by $g$.

(a) Let $H$ be the subgroup generated by $g^{21}$. How many elements does $H$ have?
*Solution.* $H = \langle g^{21} \rangle = \langle g^{(21,30)} \rangle = \langle g^3 \rangle$. *The element* $g^3$ *has order* $30/3 = 10$, *so* $H$ *has order* 10.

(b) How many subgroups does $G$ have?
*Solution. Subgroups are of the form* $\langle g^d \rangle$ *where* $d$ *divides* 30. *Since* $30 = 2 \cdot 3 \cdot 5$, *divisors are of the form* $2^a 3^b 5^c$ *with* $a, b, c \in \{0, 1\}$. *So there are* $2 \cdot 2 \cdot 2$ *possibilities for* $a, b, c$ *giving the divisors* $1, 2, 3, 6, 5, 10, 15, 30$. *So there are* 8 *subgroups.*

(c) How many elements does the automorphism group $\mathrm{Aut}(G)$ have?
*Solution.* $\mathrm{Aut}(G)$ *is isomorphic to the multiplicative group*

$$(\mathbb{Z}_{30}^{\times}, \cdot) = \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

*This group has* $\phi(30) = \phi(2 \cdot 3 \cdot 5) = (2-1)(3-1)(5-1) = 8$ *elements.*

(2) Assume that $G$ is a finite group acting on a finite set $S$.

(a) Suppose that $|G| > |S|$. Show that for every $x \in S$, the stabilizer group $G_x$ is nontrivial.
*Solution. For* $x \in S$ *we have* $|G|/|G_x| = [G : G_x] = |G \cdot x| \leq |S| < |G|$, *because* $G \cdot x \subseteq S$. *It follows that* $|G_x| > 1$, *so* $G_x$ *is nontrivial.*

(b) Suppose that $G$ is a $p$-group and $p$ does not divide $|S|$. Show that $G_x = G$ for some $x \in S$.
*Solution. We have a theorem that states that* $|S| \equiv |S^G| \bmod p$ *when* $G$ *is a finite* $p$*-group acting on a finite set* $S$. *Since* $|S|$ *is not divisible by* $p$, *neither is* $|S^G|$. *In particular* $|S^G|$ *is nonzero and there exists an element* $x \in S^G$. *Then* $x$ *is a fixed point and* $G_x = G$.

(3) Let $G$ be a group with center $Z(G)$, and let $H$ be a subgroup of $G$.

(a) Show that if $H \subseteq Z(G)$, then $H$ is normal in $G$.
*Solution. Suppose* $g \in G$ *and* $h \in H$. *Because* $h \in H \subseteq Z(G)$, *we have* $gh = hg$ *and* $ghg^{-1} = hgg^{-1} = h \in H$. *This shows that* $ghg^{-1} \in H$ *for all* $g \in G$ *and* $h \in H$, *which means that* $H$ *is a normal subgroup of* $G$.

(b) Show that if $H \subseteq Z(G)$ and $G/H$ is cyclic, then $G$ is abelian.
*Solution.* *Suppose that $G/H$ is generated by the coset $gH \in G/H$. Suppose $a \in G$. Then $aH = (gH)^k = g^k H$ for some $k \in \mathbb{Z}$. Since $a \in g^k H$ we can write $a = g^k h$ for some $h \in H$. If $b$ is another element of $G$ then it is of the form $b = g^\ell u$ for some $\ell \in \mathbb{Z}$ and $u \in H$. Now $ab = g^k h g^\ell u = g^k g^\ell h u = g^\ell g^k u h = g^\ell u g^k h = ba$ because $h$ and $u$ lie in the center of $G$. So $G$ is abelian.*

(4) Let $G$ be the group of $3 \times 3$ upper-diagonal matrices with entries in $\mathbb{Z}_2$ and 1's down the diagonal:

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z}_2 \right\}.$$

(a) Show that $G$ is a non-abelian group of order 8.
*Solution.* *$G$ is a subset $\mathrm{GL}_2(\mathbb{Z}_2)$. As a set, $G = \mathbb{Z}_2^3$ (thus, $G$ has order 8), with multiplication of matrices in $\mathrm{GL}_2(\mathbb{Z}_2)$ inducing the operation $(a, b, c) \cdot (a', b', c') = (a+a', b+b', c+c'+ab')$, with the output belonging to $G$. Since $\mathrm{GL}_2(\mathbb{Z}_2)$ is a finite group and $G$ is a subset closed under multiplication, $G$ is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_2)$. Clearly, $(1, 0, 0) \cdot (0, 1, 0) = (1, 1, 1)$ is different from $(0, 1, 0) \cdot (1, 0, 0) = (1, 1, 0)$, and so $G$ is not abelian.*

(b) What is the center of $G$?
*Solution.* *The only elements $(a, b, c) \in G$ that commute with all $(a', b', c') \in G$ are those for which $ab' = a'b$ for all $a', b' \in \mathbb{Z}_2$. Taking $a' = 0, b' = 1$ gives $a = 0$; taking $a' = 1, b' = 0$ gives $b = 0$. Thus, $Z(G)$ consists of all elements of the form $(0, 0, c)$, that is, $Z(G) = \{(0, 0, 0), (0, 0, 1)\} \cong \mathbb{Z}_2$.*

(c) Up to isomorphism, there are only two non-abelian groups of order 8, namely, the dihedral group $D_4$ and the quaternion group $Q_8$. Is the group $G$ isomorphic to $D_4$ or to $Q_8$? Explain.
*Solution.* *The group $G$ is not isomorphic to $Q_8$, since the orders of their elements don't match. For instance, $G$ has five elements of order 2 (namely, $(0, 0, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$), whereas $Q_8 = \{\pm 1, \pm i, \pm j \pm k\}$ has only one, namely, $-1$. Since we know that $G$ is a non-abelian group of order 8, it follows that $G$ must be isomorphic to $D_4$.*
*Alternatively, we can define an explicit isomorphism from $G = \langle (1, 1, 1), (1, 0, 0) \rangle$ to $D_4 = \langle a, b \mid a^4 = b^2 = 1, ba = a^{-1} b \rangle$ by sending $(1, 1, 1)$ to $a$ and $(1, 0, 0)$ to $b$; clearly, the relations among the two sets of generators match under this correspondence.*

(5) Let $S = \mathbb{R} \setminus \{0, 1\}$. Define the functions $f$ and $g$ from $S$ to $S$ by $f(x) = 1/x$ and $g(x) = (x - 1)/x$.

(a) Show that $f$ and $g$ are one-to-one and onto and find the inverse functions $f^{-1}$ and $g^{-1}$.
*Solution.* *If $x \in S$, then $f(f(x)) = 1/(1/x) = x$ so the inverse function of $f(x)$ is $f(x)$ itself. If $y = (x - 1)/x$ for $x \in S$, then $yx = x - 1$ and $x = 1/(1 - y)$. We have $g(x) = 1 - 1/x$. Let $h(x) = 1/(1 - x)$. Then $h$ is a function from $S$ to*

*S and we have*

$$h(g(x)) = \frac{1}{1 - \frac{x-1}{x}} = \frac{1}{\frac{1}{x}} = x$$

*and* $g(h(x)) = 1 - \frac{1}{h(x))} = 1 - (1-x) = x$. *So* $h$ *is the inverse function of* $g$.

(b) Show that the subgroup of $\mathrm{Sym}(S)$ generated by $f$ and $g$ is isomorphic to the symmetric group $S_3$.
*Solution. Let* $e = \mathrm{id}_S$ *be the identity function of* $S$*, that is, the identity of* $\mathrm{Sym}(S)$. *Note that* $f^2 = e$ *and* $g^2 \neq e$*, yet* $g^3 = e$*; moreover,* $fgf = g^2$. *Thus,* $G = \langle f, g \rangle = \{e, f, g, g^2, fg, fg^2\}$. *We can define a homomorphism* $\phi \colon G \to S_3$ *by* $\phi(f) = (12)$ *and* $\phi(g) = (123)$*; clearly,* $\phi$ *respects the relations in both groups, and so it is well-defined. Moreover, it is a bijection, and hence an isomorphism. Therefore, the group* $G = \langle f, g \rangle$ *is isomorphic to* $S_3$.

(6) Suppose that $G$ is a group of order $n$ and $H$ is a group of order $m$. (A group homomorphism $\phi \colon G \to H$ is trivial if $\phi(g) = e$ for all $g \in G$.)

(a) Suppose that $(m, n) = 1$. Show that every group homomorphism $\phi \colon G \to H$ must be trivial. (Hint: Consider the order of $\phi(g)$ for $g \in G$.)

*Solution. Suppose* $g \in G$. *The order of* $\phi(g) \in H$ *divides* $|H| = m$. *Since* $g \in G$*, and* $n = |G|$ *we have* $g^n = e$ *and* $\phi(g)^n = \phi(g^n) = \phi(e) = e$. *So the order of* $\phi(g)$ *divides* $n$. *Now the order of* $\phi(g)$ *also divides* $(m, n) = 1$ *so the order is* 1 *and* $\phi(g) = e$.

(b) Suppose that $G$ is cyclic and $(m, n) \neq 1$. Prove that there exists a nontrivial group homomorphism $\phi \colon G \to H$. (Hint: There exists a prime $p$ that divides $m$ and $n$.)
*Solution. Let* $p$ *a common prime factor of* $m$ *and* $n$. *By Cauchy's theorem* $H$ *has an element* $h$ *of order* $p$ *because* $p$ *is a prime that divides* $|H|$. *Suppose* $G = \langle g \rangle$. *Then we can define a group homomorphism* $\phi \colon G \to H$ *by* $\phi(g^k) = h^k$. *This is well-defined because the order of* $h$ *divides the order of* $g$.

(7) Let $G$ be a group of order $5 \cdot 7 \cdot 11 = 385$.

(a) Show that $G$ has a normal subgroup of order 7.
*Solution. Let* $k_p$ *be the number of Sylow* $p$*-subgroups of* $G$. *Then* $k_7$ *divides* $5 \cdot 11 = 55$. *This means that* $k_7 \in \{1, 5, 11, 55\}$. *We also have* $k_7 \equiv 1 \bmod 7$. *So* $k_7$ *has to be* 1. *This means that the* 7*-Sylow subgroup, which has* 7 *elements, is normal.*

(b) Suppose that $G$ does not have a normal subgroup of order 5. How many subgroups of order 5 does $G$ have? How many elements of order 5 does $G$ have?
*Solution. Since* $k_5$ *divides* $77 = 7 \cdot 11$ *we have* $k_5 \in \{1, 7, 11, 77\}$. *Because* $k_5 \equiv 1 \bmod 5$ *we have* $k_5 = 1$ *or* $k_5 = 11$. *Since we assume there is no normal*

*subgroup with* $5$ *elements, we get* $k_5 = 11$*. So there are 11 subgroups with* $5$ *elements. Each of these subgroups has* $4$ *elements of order* $5$*, so there are* $11 \cdot 4 = 44$ *elements of order 5.*

(8) Let $\sigma = (1\ 5\ 3\ 2)(2\ 6\ 3\ 7) \in S_8$.

(a) Write $\sigma$ as a product of disjoint cycles.
*Solution.* $\sigma = (1\ 5\ 3\ 7)(2\ 6)$

(b) Write $\sigma$ as a product of transpositions. Is $\sigma$ an even or odd permutation?
*Solution.* $\sigma = (1\ 5)(5\ 3)(3\ 7)(2\ 6)$*. But there are many other ways, such as* $\sigma = (3\ 7)(5\ 7)(1\ 7)(2\ 6)$.

(c) What is $\sigma^{50}$?
*Solution,* $\sigma^{50} = (1\ 5\ 3\ 7)^{50}(2\ 6)^{50} = (1\ 5\ 3\ 7)^2 = (1\ 3)(5\ 7)$.

(9) The symmetric group $G = S_4$ acts on the set $X = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ by $\sigma \cdot (i, j) = (\sigma(i), \sigma(j))$ for every $(i, j) \in X$ and $\sigma \in S_4$.

(a) Describe the orbits of $G$ in $X$.
*Solution, one orbit* $G \cdot (1, 1) = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ *and another orbit is* $G \cdot (1, 2) = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$.

(b) What is the stabilizer group $G_{(1,2)}$ of $(1, 2) \in X$?
*Solution.* $\sigma \cdot (1, 2) = (\sigma(1), \sigma(2))$ *is equal to* $(1, 2)$ *if and only if* $\sigma(1) = 1$ *and* $\sigma(2) = 2$*. The stabilizer is equal to* $G_{(1,2)} = \{1, (3\ 4)\} \subseteq S_4$.

(10) Suppose that $(G, \cdot)$ is a group, and consider the subgroup $N = \{(g, g) \mid g \in G\}$ of $G \times G$.

(a) Show that $N$ is a normal subgroup of $G \times G$ if and only if $G$ is abelian.
*Solution. If* $G$ *is abelian, tnen* $G \times G$ *is abelian and every subgroup, in particular also* $N$*, is normal. Conversely, suppose that* $N$ *is normal. Suppose* $g, h \in G$*. Then* $(h, h) \in N$ *and* $(g, e) \in G \times G$*. So* $(g, e)(h, h)(g, e)^{-1} = (ghg^{-1}, h) \in N$*. This implies* $ghg^{-1} = h$ *and* $gh = hg$*. This proves that* $G$ *is abelian.*

(b) Suppose that $G$ is abelian. Show that $(G \times G)/N$ is isomorphic to $G$.
*Solution. Define a function homomorphism* $\phi\colon G \times G \to G$ *by* $\phi(g_1, g_2) = g_1 g_2^{-1}$*. Then* $\phi$ *is a group homomorphism:* $\phi((g_1, g_2)(h_1, h_2)) = \phi(g_1 h_1, g_2 h_2) = g_1 h_1 (g_2 h_2)^{-1} = g_1 h_1 h_2^{-1} g_2^{-1} = g_1 g_2^{-1} h_1 h_2^{-1} = \phi(g_1, g_2)\phi(h_1, h_2)$*. Since* $\phi(g, e) = g$ *we see that* $\phi(G \times G) = G$*. An element* $(g_1, g_2)$ *lies in the kernel of* $\phi$ *when* $g_1 g_2^{-1} = e$*. This proves that the kernel of* $\phi$ *is exactly* $N$*. By the fundamental homomorphism theorem,* $G \times G/N = G \times G/\ker(\phi) \cong \phi(G \times G) = G$.