Name: **Section:**

1	2	3	4	5	6	7	8	9	10	total

FINAL EXAM (120 MIN)
MATH 3175 GROUP THEORY
SPRING 2024

Instructions: One calculator is allowed but no other electronics. One letter-size, single sided crib sheet is allowed, but no other notes. Each of the 10 problems is worth 10 points.

(1) Suppose $G = \langle g \rangle$ is a cyclic group of order 30 generated by g.

(a) Let H be the subgroup generated by g^{21}. How many elements does H have?

(b) How many subgroups does G have?

(c) How many elements does the automorphism group $\mathrm{Aut}(G)$ have?

(2) Assume that G is a finite group acting on a finite set S.

(a) Suppose that $|G| > |S|$. Show that for every $x \in S$, the stabilizer group G_x is nontrivial.

(b) Suppose that G is a p-group and p does not divide $|S|$. Show that $G_x = G$ for some $x \in S$.

(3) Let G be a group with center $Z(G)$, and let H be a subgroup of G.

(a) Show that if $H \subseteq Z(G)$, then H is normal in G.

(b) Show that if $H \subseteq Z(G)$ and G/H is cyclic, then G is abelian.

(4) Let G be the group of 3×3 upper-diagonal matrices with entries in $\mathbb{Z}_2$ and 1's down the diagonal:

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z}_2 \right\}.$$

(a) Show that G is a non-abelian group of order 8.

(b) What is the center of G?

(c) Up to isomorphism, there are only two non-abelian groups of order 8, namely, the dihedral group D_4 and the quaternion group Q_8. Is the group G isomorphic to D_4 or to Q_8? Explain.

(5) Let $S = \mathbb{R} \setminus \{0, 1\}$. Define the functions f and g from S to S by $f(x) = 1/x$ and $g(x) = (x-1)/x$.

(a) Show that f and g are one-to-one and onto and find the inverse functions f^{-1} and g^{-1}.

(b) Show that the subgroup of $\text{Sym}(S)$ generated by f and g is isomorphic to the symmetric group S_3.

(6) Suppose that G is a group of order n and H is a group of order m. (A group homomorphism $\phi\colon G \to H$ is trivial if $\phi(g) = e$ for all $g \in G$.)

   (a) Suppose that $(m, n) = 1$. Show that every group homomorphism $\phi\colon G \to H$ must be trivial. (Hint: Consider the order of $\phi(g)$ for $g \in G$.)

   (b) Suppose that G is cyclic and $(m, n) \neq 1$. Prove that there exists a nontrivial group homomorphism $\phi\colon G \to H$. (Hint: There exists a prime p that divides m and n.)

(7) Let G be a group of order $5 \cdot 7 \cdot 11 = 385$.

(a) Show that G has a normal subgroup of order 7.

(b) Suppose that G does not have a normal subgroup of order 5. How many subgroups of order 5 does G have? How many elements of order 5 does G have?

(8) Let $\sigma = (1\ 5\ 3\ 2)(2\ 6\ 3\ 7) \in S_8$.

(a) Write σ as a product of disjoint cycles.

(b) Write σ as a product of transpositions. Is σ an even or odd permutation?

(c) What is σ^{50}?

(9) The symmetric group $G = S_4$ acts on the set $X = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ by $\sigma \cdot (i, j) = (\sigma(i), \sigma(j))$ for every $(i, j) \in X$ and $\sigma \in S_4$.

(a) Describe the orbits of G in X.

(b) What is the stabilizer group $G_{(1,2)}$ of $(1, 2) \in X$?

(10) Suppose that $(G, \cdot)$ is a group, and consider the subgroup $N = \{(g, g) \mid g \in G\}$ of $G \times G$.

(a) Show that N is a normal subgroup of $G \times G$ if and only if G is abelian.

(b) Suppose that G is abelian. Show that $(G \times G)/N$ is isomorphic to G.