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NORTHEASTERN UNIVERSITY
DEPARTMENT OF MATHEMATICS

MATH 3150 **Real Analysis** **Spring 2011**

Final Exam

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1. Let $f_1, \ldots, f_k$ be functions defined on a subset $A \subseteq \mathbb{R}^n$ and taking values in $\mathbb{R}$. Let $f = \sum_{i=1}^{k} f_i$, and set

$$m_i = \inf\{f_i(x) \mid x \in A\}, \qquad m = \inf\{f(x) \mid x \in A\},$$

(a) Show that $m \geq \sum_{i=1}^{k} m_i$.

(b) Given an example where equality fails.

2. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be two subsets, and consider their product, $A \times B$, viewed as a subset in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$.

(a) Suppose A and B are path-connected. Show that $A \times B$ is path-connected.

(b) Suppose A and B are bounded. Show that $A \times B$ is bounded.

3. Consider the following subset of the plane:

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^4 = 1\}.$$

Show that A is compact.

4. Consider the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x = 1 - \dfrac{1}{n}, \text{ for some integer } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is Riemann-integrable, and that $\displaystyle\int_0^1 f(x)dx = 0$.

5. Consider the function $f\colon [0,\pi] \to \mathbb{R}$ given by

$$f(x) = \int_0^{x^2} \cos\left(\sqrt{t}\,\right) dt$$

(a) What is $f(0)$?

(b) Show that f is differentiable. What is its derivative?

(c) When $x = \pi/3$, show that $f'(x) = x$.

6. Let $f\colon [0,+\infty) \to \mathbb{R}$ be a continuous function, differentiable on $(0,\infty)$. Suppose that

$$f(x) + x \cdot f'(x) \geq 0, \quad \text{for all } x > 0.$$

Show that $f(x) \geq 0$, for all $x \geq 0$.

7. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ and $u, v, w\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable functions, and let $F\colon \mathbb{R}^3 \to \mathbb{R}$ be the function given by $F(x, y, z) = f(u(x, z), v(x, y), w(y, z))$.

(a) Use the Chain Rule to express $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$ in terms of the partial derivatives of f, u, v, and w.

(b) Now suppose

$$f(x, y, z) = x^3 - x^2y^2 + z^4, \quad u(x, y) = x + y, \quad v(x, y) = 2xy, \quad w(x, y) = x^2 + y^3.$$

Compute $\partial F/\partial x$, $\partial F/\partial y$, and $\partial F/\partial z$, either using part (a), or directly (or both).

8. Consider the surface S in $\mathbb{R}^3$ given by the equation $x^2 + yz - xz = 4$.

(a) Find a unit normal vector to S at the point $(1, 2, 3)$.

(b) Find the equation of the tangent plane to the surface S at the point $(1, 2, 3)$.