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**MATH 3150** | **Real Analysis** | **Fall 2016**

# Solutions for the Midterm Exam

**Instructions**: Write your name in the space provided. Calculators are permitted, but books, notes, or laptops are **not** allowed. Each problem is worth 12 points.

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**1.** Complete the definitions in (a), (b) and (d). Prove (c).

(a) A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converges to $x \in \mathbb{R}$ if:

For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$, for all $n \geq N$.

(b) A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is a Cauchy sequence if:

For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$, for all $m, n \geq N$.

(c) In $\mathbb{R}$, show that if $\{x_n\}_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$, then it is a Cauchy sequence.

Let $\varepsilon > 0$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x$, there exists an $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon/2$, for all $n \geq N$. By the triangle inequality,

$$|x_m - x_n| \leq |x_m - x| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for all $m, n \geq N$. Thus, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

(d) A metric space $(X, \rho)$ is complete if:

Every Cauchy sequence in $X$ converges to some point $x \in X$.

**2.** Define a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ recursively by setting $x_0 = 0$ and $x_n = \dfrac{x_{n-1}^2 + 2}{3}$ for $n \geq 1$.

(a) Show by induction that $x_n$ is monotonically increasing.

Starting at $n = 0$, we see that $x_0 = 0 \leq 2/3 = x_1$. For the induction step, assume $0 \leq x_{n-1} \leq x_n$. Then $x_{n-1}^2 \leq x_n^2$, and so

$$0 \leq x_n = \frac{x_{n-1}^2 + 2}{3} \leq \frac{x_n^2 + 2}{3} = x_{n+1}.$$

This completes the induction.

(b) Show by induction that $x_n$ is bounded above by 1.

Starting at $n = 0$, we see that $x_0 = 0 \leq 1$. For the induction step, assume $x_{n-1} \leq 1$. Then

$$x_n = \frac{x_{n-1}^2 + 2}{3} \leq \frac{1+2}{3} = 1.$$

This completes the induction.

(c) Prove that $x_n$ converges and compute $\lim_{n\to\infty} x_n$.

By parts (a) and (b), the sequence $\{x_n\}$ is monotonically increasing and bounded above (by 1). Therefore, $\{x_n\}$ converges. Denote the limit of this sequence by $x$. Taking the limit as $n \to \infty$ on both sides of the recursion formula $x_n = \frac{x_{n-1}^2+2}{3}$, we obtain that $x = \frac{x^2+2}{3}$, or, $x^2 - 3x + 2 = 0$. This equation has two solutions, $x = 1$ and $x = 2$, but only the first one is valid, since $x_n \leq 1$ implies $\lim_{n\to\infty} x_n \leq 1$. Hence,

$$\lim_{n\to\infty} x_n = 1.$$

**3.** Let $A$ be the subset of $\mathbb{R}$ given by $A = ([0,1]\backslash\mathbb{Q}) \cup (1,2) \cup \{3\}$.

(a) Find the interior $A^\circ$ of $A$ and the closure of the interior $\overline{A^\circ}$ in $\mathbb{R}$.

$$\begin{aligned} A^\circ &= (1,2) \\ \overline{A^\circ} &= [1,2] \end{aligned}$$

(b) Find the closure $\overline{A}$ of $A$ and the interior of the closure $(\overline{A})^\circ$ in $\mathbb{R}$.

$$\begin{aligned} \overline{A} &= [0,2] \cup \{3\} \\ (\overline{A})^\circ &= (0,2) \end{aligned}$$

(c) Find the boundary of $A$.

$$\begin{aligned} \partial A &= \overline{A} \setminus A^\circ \\ &= \Big([0,2] \cup \{3\}\Big) \setminus (1,2) \\ &= [0,1] \cup \{2,3\} \end{aligned}$$

(d) Find the closure of the complement $\overline{A^c}$ of $A$ in $\mathbb{R}$.

$$\begin{aligned} A^c &= (-\infty,0) \cup ([0,1] \cap \mathbb{Q}) \cup [2,3) \cup (3,\infty) \\ \overline{A^c} &= (-\infty,1] \cup [2,\infty) \end{aligned}$$

**4.** (a) Does the series $\sum_{n=0}^{\infty} \frac{n^{10}}{10^n}$ converge or not? Indicate a reason, or which test is used and how.

Use the Ratio Test:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}} \\ &= \frac{(n+1)^{10}}{n^{10}} \cdot \frac{10^n}{10^{n+1}} \\ &= \frac{1}{10} \cdot \left(1 + \frac{1}{n}\right)^{10} \\ &\xrightarrow[n\to\infty]{} \frac{1}{10} \cdot (1+0)^{10} \\ &= \frac{1}{10} < 1.\end{aligned}$$

Hence, the series converges.

(b) Does the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ converge? If yes, **compute** the series; if not, indicate a reason.

Using partial fractions, we see that

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right).$$

This allows us to compute explicitly the partial sums of this telescoping series:

$$\begin{aligned}s_k &= \sum_{n=1}^{k} \frac{1}{(2n-1)(2n+1)} \\ &= \sum_{n=1}^{k} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{2k-1} - \frac{1}{2k+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2k+1}\right)\end{aligned}$$

Hence, $\lim_{k\to\infty} s_k = \frac{1}{2}$, and the series converges. The sum of the series, then, is the limit of the partial sums,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$$

**5.** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a complete metric space $(X, \rho)$.

(a) Suppose that $\rho(x_{n+1}, x_n) \leq \alpha\rho(x_n, x_{n-1})$ for all $n \geq 2$, where $0 < \alpha < 1$. Show that $\{x_n\}$ converges.

Since the metric space $X$ is complete, it suffices to show that $\{x_n\}$ is a Cauchy sequence. To start with let us note that, for all $n \geq 1$,

$$\rho(x_{n+1}, x_n) \leq \alpha^{n-1}\rho(x_2, x_1),$$

This claim is proved by induction on $n$. For $n = 1$, the claim simply says $\rho(x_2, x_1) \leq \alpha^0\rho(x_2, x_1)$, which is obviously true, since $\alpha^0 = 1$. Assuming the claim holds for $n$, let us verify it holds for $n + 1$:

$$\rho(x_{n+2}, x_1) \leq \alpha\rho(x_{n+1}, x_n) \leq \alpha\alpha^{n-1}\rho(x_2, x_1) = \alpha^n\rho(x_2, x_1),$$

and so the induction step is verified.

Now, for every $m > n$, the triangle inequality, the above formula, and the fact that $0 < \alpha < 1$ imply that

$$\begin{aligned}
\rho(x_m, x_n) &\leq \rho(x_m, x_{m-1}) + \rho(x_{m-1}, x_{m-2}) + \cdots + \rho(x_{n+1}, x_n) \\
&\leq \alpha^{m-2}\rho(x_2, x_1) + \alpha^{m-3}\rho(x_2, x_1) + \cdots + \alpha^{n-1}\rho(x_2, x_1) \\
&= \alpha^{n-1}\rho(x_2, x_1)\big(\alpha^{m-n-1} + \alpha^{m-n-2} + \cdots + \alpha + 1\big) \\
&\leq \alpha^{n-1}\rho(x_2, x_1)\sum_{k=0}^{\infty}\alpha^k \\
&= \alpha^{n-1}\rho(x_2, x_1)\frac{1}{1-\alpha}
\end{aligned}$$

But $\lim\limits_{n\to\infty} \alpha^{n-1} = 0$, and thus $\lim\limits_{n\to\infty} \alpha^{n-1}\rho(x_2, x_1)\frac{1}{1-\alpha} = 0$, too. Therefore, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\rho(x_m, x_n) < \varepsilon$ for all $m > n \geq N$, thereby showing that $\{x_n\}$ is a Cauchy sequence, and thus, a convergent sequence.

(b) Suppose instead that $\rho(x_{n+1}, x_n) \leq \dfrac{1}{\sqrt{n}}$ for all $n \geq 1$. Show by means of an example that $\{x_n\}$ may **not** converge.

Let $X = \mathbb{R}$ with the usual metric $\rho(x, y) = |x - y|$, and let $x_n = \displaystyle\sum_{k=1}^{n-1}\frac{1}{\sqrt{k}}$. Then

$$\rho(x_{n+1}, x_n) = |x_{n+1} - x_n| = \frac{1}{\sqrt{n}}$$

for all $n \geq 1$, as required. On the other hand, $x_n$ is the $(n-1)$-th partial sum of $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}$, which is a $p$-series with $p = \frac{1}{2}$, and thus a divergent series. Hence, the sequence $\{x_n\}$ does not converge.

**6.** In a metric space $(X, \rho)$, let $A \neq \emptyset$ be a subset of $X$.

(a) Define the concept "an element $x \in X$ is *a limit point* of $A$".

We say that $x$ is a limit point of $A$ if every neighborhood $U$ of $x$ contains some element of $A$ different from $x$. In other words

$$(U \setminus \{x\}) \cap A \neq \emptyset,$$

or, equivalently, there is a $y \in U \cap A$ such that $y \neq x$.

Note: The neighborhood $U$ can be replaced by an arbitrary open ball $B_r(x)$.

(b) Define the concepts "a subset $U \subset X$ is *open* in $X$" and "a subset $F \subset X$ is *closed* in $X$".

A subset $U \subset X$ is open in $X$ if for every $x \in X$, there is an $r > 0$ such that $B_r(x) \subset U$.

A subset $F \subset X$ is closed in $X$ if its complement $F^c$ is open in $X$.

(c) Let $A'$ denote the set of limit points of $A$ in $X$. Show that $A'$ is closed in $X$.

If $A'$ is empty, then obviously $A'$ is closed. So let us assume $A' \neq \emptyset$. We need to show

$$\overline{A'} \subseteq A'.$$

Let $x \in \overline{A'}$, and let $U$ be an (open) neighborhood of $x$. Then, since $x$ is in the closure of $A'$,

$$U \cap A' \neq \emptyset.$$

Let $y \in U \cap A'$. There are two cases to consider:

(1) If $y = x$, then $x \in A'$ and we are done.

(2) If $y \neq x$, then $U \setminus \{x\}$ is a neighborhood of $y$, since recall every singleton in a metric space is a closed subset, and thus $U \setminus \{x\} = (X \setminus \{x\}) \cap U$ is open. Hence,

$$(U \setminus \{x, y\}) \cap A \neq \emptyset,$$

since $y$ belongs to $A'$. But this implies

$$(U \setminus \{x\}) \cap A \neq \emptyset,$$

and since $U$ was an arbitrary neighborhood of $x$, we conclude that $x \in A'$.