Name: ______________________

Professor Alex Suciu

MATH 3150 **Real Analysis** **Fall 2016**

Midterm Exam

Instructions: Write your name in the space provided. Calculators are permitted, but books, notes, or laptops are **not** allowed. Each problem is worth 12 points.

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1. Complete the definitions in (a), (b) and (d). Prove (c).

(a) A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converges to $x \in \mathbb{R}$ if:

(b) A sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ is a Cauchy sequence if:

(c) In $\mathbb{R}$, show that if $\{x_n\}_{n=1}^{\infty}$ converges to $x \in \mathbb{R}$ then it is a Cauchy sequence.

(d) A metric space (X, ρ) is complete if:

2. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ recursively by setting $x_0 = 0$ and $x_n = \dfrac{x_{n-1}^2 + 2}{3}$ for $n \geq 1$.

(a) Show by induction that x_n is monotonically increasing.

(b) Show by induction that x_n is bounded above by 1.

(c) Prove that x_n converges and compute $\lim\limits_{n\to\infty} x_n$.

3. Let A be the subset of $\mathbb{R}$ given by $A = ([0,1]\backslash\mathbb{Q}) \cup (1,2) \cup \{3\}$.

(a) Find the interior A° of A and the closure of the interior $\overline{A^\circ}$ in $\mathbb{R}$.

(b) Find the closure $\overline{A}$ of A and the interior of the closure $(\overline{A})^\circ$ in $\mathbb{R}$.

(c) Find the boundary of A.

(d) Find the closure of the complement $\overline{A^c}$ of A in $\mathbb{R}$.

4. (a) Does the series $\sum_{n=0}^{\infty} \frac{n^{10}}{10^n}$ converge or not? Indicate a reason, or which test is used and how.

(b) Does the series $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ converge? If yes, **compute** the series; if not, indicate a reason.

5. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a complete metric space (X, ρ).

(a) Suppose that $\rho(x_{n+1}, x_n) \le \alpha\rho(x_n, x_{n-1})$ for all $n \ge 2$, where $0 < \alpha < 1$. Show that $\{x_n\}$ converges.

(b) Suppose instead that $\rho(x_{n+1}, x_n) \le \dfrac{1}{\sqrt{n}}$ for all $n \ge 1$. Show by means of an example that $\{x_n\}$ may **not** converge.

6. In a metric space (X, ρ), let $A \neq \emptyset$ be a subset of X.

(a) Define the concept "an element $x \in X$ is *a limit point* of A".

(b) Define the concepts "a subset $U \subset X$ is *open* in X" and "a subset $F \subset X$ is *closed* in X".

(c) Let A' denote the set of limit points of A in X. Show that A' is closed in X.