1. (10 pts) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < \frac{1}{n^3} \quad \text{for all } n \in \mathbb{N}$$

   Prove that (s_n) is a Cauchy sequence and hence a convergent sequence.

2. Consider the sequence (x_n) with terms $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \ldots$.
   (a) (5 pts) Show that (x_n) is bounded.
   (b) (10 pts) Show directly from the definition that (x_n) is *not* a Cauchy sequence.
   (c) (5 pts) Find two convergent subsequences of (x_n) that converge to two different limits.
   (d) (5 pts) What conclusion regarding the convergence of the sequence (x_n) can you draw from part (c), and how does that conclusion compare to the answer in part (b)?

3. Let (x_n) be a sequence of real numbers. In each of the following situations, decide whether the sequence converges: if yes, give a proof why; otherwise, give an example where it does not.
   (a) (10 pts) $|x_n - x_k| \leq \frac{1}{n} + \frac{1}{k}$ for all $n, k \geq 1$.
   (b) (10 pts) For all $\epsilon > 0$, there is an $n > 1/\epsilon$ such that $|x_n| < \epsilon$.

4. Let (a_n) be a sequence and let c, d be real numbers with $c < d$. Assume that the terms in the sequence (a_n) are eventually in the closed interval $[c, d]$. Prove that
   (a) (10 pts) (a_n) is bounded

   (b) (10 pts) $\liminf s_n$ and $\limsup s_n$ are elements in $[c, d]$.

5. (10 pts) Let (a_n) be a bounded sequence, and let (a_{n_k}) be a convergent subsequence (a_n). Prove that

$$\liminf a_n \leq \lim a_{n_k} \leq \limsup a_n$$

6. For each of the following series, determine whether the series converges or diverges. Justify your answers.
   (a) (5 pts) $\sum \frac{\sin(n)}{n^2}$

   (b) (5 pts) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

   (c) (5 pts) $\sum \frac{5n^2 + 6n - 2}{3^n + 1}$