Professor Alex Suciu

MATH 3150 **Real Analysis** **Fall 2016**

Solutions for Homework 6

Problem 1. (Problem 1, page 144)

(a) Consider the function $f(x) = \sum_{n=1}^{\infty} x^n$.

(1) This is a power series with radius of convergence $R = 1$. Thus, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Clearly, it also diverges when $x = \pm 1$.

(2) By the general theory, the series converges uniformly on all intervals $[-a, a]$ with $0 < a < 1$. On the other hand, it does *not* converge uniformly on the interval $(-1, 1)$, since its sequence of partial sums, $s_k(x) = \sum_{n=1}^{k} x^n$, is not uniformly Cauchy on $(-1, 1)$:

$$\sup\{|s_k(x) - s_{k-1}(x)| \mid x \in (-1,1)\} = \sup\{|x|^k \mid x \in (-1,1)\} = 1.$$

(3) Also by the general theory, the sum of the series is a continuous function on any subset $S \subset \mathbb{R}$ on which the series converges uniformly. Thus, the function $f(x) = \frac{x}{1-x}$ is continuous on all intervals $[-a, a]$ with $0 < a < 1$.

(b) Consider the function $f(x) = \sum_{n=1}^{\infty} \frac{1}{1-x^n}$.

(1) The series converges absolutely for $|x| > 1$, since

$$\lim_{n\to\infty} \frac{\left|\frac{1}{1-x^n}\right|}{\left|\frac{1}{x^n}\right|} = \lim_{n\to\infty} \left|\frac{x^n}{1-x^n}\right| = 1,$$

and the claim follows by the comparison test with the geometric series of ratio $\frac{1}{|x|}$. Furthermore series diverges for $|x| \le 1$, since in this case the sequence $\frac{1}{1-x^n}$ has limit 1 if $|x| < 1$, or fails to converge if $x = \pm 1$.

(2) The series converges uniformly on all intervals of the form $(\infty, -a]$ and $[a, \infty)$ with $a > 1$, by the Weierstrass M-test. Indeed, on those intervals,

$$\left|\frac{1}{1-x^n}\right| \le \frac{1}{|x|^n - 1} \le \frac{1}{a^n - 1},$$

and the series $\sum_{n=1}^{\infty} \frac{1}{a^n-1}$ again converges by comparison with the geometric series of ratio $1/a$.

On the other hand, the series does *not* converge uniformly on the intervals $(\infty, -1)$ and $(1, \infty)$, since its sequence of partial sums, is not uniformly Cauchy on those intervals, e.g.,

$$\sup\{|s_k(x) - s_{k-1}(x)| \mid x > 1\} = \sup\{1/|1 - x^k| \mid x > 1\} = \infty.$$

(3) The function f is continuous on all intervals of the form $(\infty, -a]$ and $[a, \infty)$ with $a > 1$.

Problem 2. (Problem 2, page 144) Let $f_n(x) = x + 1/n$ and $f(x) = x$, for $x \in \mathbb{R}$.

(a) $f_n \to f$ uniformly on $\mathbb{R}$, since
$$\sup\{|f_n(x) - f(x)| \mid x \in \mathbb{R}\} = 1/n \to 0.$$

(b) f_n^2 does *not* converge uniformly to f on $\mathbb{R}$ (although $f_n^2 \to f^2$ pointwise, since, if $f_n \to f$ and $g_n \to g$, then always $f_n g_n \to fg$). Indeed,
$$\sup\{\left|f_n^2(x) - f^2(x)\right| \mid x \in \mathbb{R}\} = \sup\{2x + 1/n^2 \mid x \in \mathbb{R}\} = \infty.$$

Problem 3. (Problem 4, page 144) Let $f_n(x) = nxe^{-nx^2}$.

(a) $f_n \to 0$ pointwise on $\mathbb{R}$. This is clear for $x = 0$ (since $f_n(0) = 0$), while for $x \neq 0$, l'Hospital's rule gives
$$\lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{x}{x^2 e^{nx^2}} = \frac{1}{x} \lim_{n\to\infty} \frac{1}{(e^{x^2})^n} = 0 \quad (\text{since } e^{x^2} > 1).$$

(b) f_n does *not* converge uniformly to 0 on any interval containing 0. Indeed,
$$f_n(1/\sqrt{n}) = \sqrt{n}/e \xrightarrow{n\to\infty} \infty.$$
Since any interval containing 0 must contain $1/\sqrt{n}$ for some large enough n, we conclude that $\|f_n\| \to \infty$.

(c) On the other hand, f_n does converge uniformly to 0 on any interval of the form $[a, \infty)$ with $a > 0$. Indeed, since $e^{nx^2} = 1 + nx^2 + \frac{1}{2}n^2x^4 + \cdots$,
$$|f_n(x)| = \frac{nx}{e^{nx^2}} \leq \frac{2nx}{n^2x^4} = \frac{2}{nx^3} \leq \frac{2}{na^3},$$
and thus
$$\lim_{n\to\infty} \sup\{|f_n(x)| \mid x \in [a, \infty)\} = 0.$$

Problem 4. (Example (3.111)-(d), page 144) Let $f_n \colon [0, 2] \to \mathbb{R}$ be defined by
$$f_n(x) = \begin{cases} 0 & \text{if } 2/n < x \leq 2, \\ n^2x^2 - 2nx & \text{if } 0 \leq x \leq 2/n. \end{cases}$$

Note that $f_n \to 0$ pointwise, since clearly $f_n(0) = 0$ for all n, and, for all $x \in (0, 2]$, there is an $n \in \mathbb{N}$ such that $x > 2/n$, and thus $f_m(x) = 0$ for all $m \geq n$.

On the other hand, f_n does not converge uniformly to 0. Indeed, note that $f_n(1/n) = -1$ for all $n \in \mathbb{N}$, and thus
$$\sup\{|f_n(x)| \mid x \in [0, 2]\} \geq 1.$$

Problem 5. (Problem 4, page 183) Let $f \colon [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Suppose $\lim_{x\to a^+} f'(x) = A$. We need to show that $f'_+(a)$ exists and is equal to A.

Let $\epsilon > 0$. From the hypothesis, there is a $\delta > 0$ such that $|f'(x) - A| < \epsilon$, for all $a < x < a + \delta$. Fix such an x. Then, by the MVT, there is an x_0 with $a < x_0 < x$ such that
$$\frac{f(x) - f(a)}{x - a} = f'(x_0).$$
Hence,
$$\left|\frac{f(x) - f(a)}{x - a} - A\right| = \left|f'(x_0) - A\right| < \epsilon.$$
This shows that $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = A$, that is, $f'_+(a) = A$.