6 points for each problem. Total score: 30.

(1) (#2-a) $\lim_{n\to\infty}(\sqrt{n^2+2n}-n)$

Proof.

$$\sqrt{n^2+2n}-n=\frac{n^2+2n-n^2}{\sqrt{n^2+2n}+n}=\frac{2n}{\sqrt{n^2+2n}+n}=\frac{2}{\sqrt{1+\frac{2}{n}}+1}$$

$\frac{2}{n}\to 0$ as $n\to\infty$. Hence the answer is 1. □

(2) (#7)

Proof. First, we prove the hint: If $\frac{a}{b}\le\frac{c}{d}$ and $b,d>0$, then $ad\le bc$. Adding ab we have $ab+ad\le bc+ab$ which gives $a(b+d)\le b(c+a)$. Hence, we have $\frac{a}{b}\le\frac{a+d}{b+c}$ since both b and $b+d$ are positive. Similarly, adding cd to both sides we have $(a+c)d\le c(b+d)$ which gives $\frac{a+c}{b+d}\le\frac{c}{d}$.

Suppose $\frac{x_n}{y_n}$ is monotonically increasing. For $n=1$, since $\frac{x_1}{y_1}\le\frac{x_2}{y_2}$, using the hint we have

$$\frac{x_1}{y_1}\le\frac{x_1+x_2}{y_1+y_2}\le\frac{x_2}{y_2}.$$

Assuming that $z_n\le z_{n+1}$ for all $n\in\mathbb{N}$, we want to show $z_{n+1}\le z_{n+2}$. From $z_n\le z_{n+1}$, we have

$$z_n\le\frac{x_{n+1}}{y_{n+1}}\qquad (*).$$

This is because if not, then

$$\frac{x_{n+1}}{y_{n+1}}<z_n=\frac{x_1+\cdots+x_n}{y_1+\cdots+y_n}.$$

Apply the hint and we have

$$\frac{x_1+\cdots+x_n+x_{n+1}}{y_1+\cdots+y_n+y_{n+1}}\le z_n,$$

i.e., $z_{n+1}\le z_n$, a contradiction.

Now, using $(*)$ and the hint, we have

$$z_{n+1}=\frac{x_1+\cdots+x_n+x_{n+1}}{y_1+\cdots+y_n+y_{n+1}}\le\frac{x_{n+1}}{y_{n+1}}\le\frac{x_{n+2}}{y_{n+2}}.$$

Apply the hint again, we get

$$z_{n+1}=\frac{x_1+\cdots+x_n+x_{n+1}}{y_1+\cdots+y_n+y_{n+1}}\le\frac{x_1+\cdots+x_{n+1}+x_{n+2}}{y_1+\cdots+y_{n+1}+y_{n+2}}=z_{n+2}.$$

The decreasing case can be shown similarly. □

(3) (# 8)

Proof. First we prove the hint. Given $0<\alpha<\beta$, we have $\sqrt{\alpha\beta}/\alpha=\sqrt{\beta/\alpha}>1$ by which we have shown $\alpha<\sqrt{\alpha\beta}$. It is trivial to see $\alpha<(\alpha+\beta)/2<\beta$. It remains to show that $\sqrt{\alpha\beta}<(\alpha+\beta)/2$. This is immediate from

$$4\alpha\beta<(\alpha+\beta)^2$$

which is from

$$0 < (\alpha - \beta)^2.$$

Now, with the hint, we what to show that the sequence x_n is an alternating sequence as in (2.14) of the textbook. By induction, for $n = 1$, we immediately have

$$x_1 = a < x_3 = \sqrt{ab} < x_4 = (a+b)/2 < x_2 = b.$$

Assume that we have

$$x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}.$$

The middle inequality and the hint imply

$$x_{2n+1} < \sqrt{x_{2n+1}x_{2n+2}} < (x_{2n+1}x_{2n+2})/2 < x_{2n+2}$$

i.e.,

$$x_{2n+1} < x_{2n+3} < x_{2n+4} < x_{2n+2}$$

which completes the induction to prove the alternating property.

To show $\lim_{n\to\infty} x_{2n} - x_{2n-1} = 0$, we use induction to show that $x_{2n} - x_{2n-1} \le (b-a)/2^{n-1}$. For $n = 1$, we have $x_2 - x_1 = b - a$, which verifies the statement. Assume it is true for n, then

$$x_{2n+2} - x_{2n+1} < x_{2n+2} - x_{2n-1} = (x_{2n} - x_{2n-1})/2 \le (b-a)/(2^{n-1} \cdot 2) = (b-a)/2^n.$$

This completes the proof of the shrinking property since $(b-a)/2^n \to 0$ as $n \to \infty$.

Then by (2.14), the sequence x_n converges. ☐

(4) (#15) Compute x_{n+2}, x_{n+3} and x_{n+4} using the recursive equation $x_{n+1} = 2 - 2/x_n$ we have

$$x_{n+2} = \frac{x_n - 2}{x_n - 1},\ x_{n+3} = -\frac{2}{x_n - 2},\ x_{n+4} = x_n.$$

Therefore, the sequence repeats the cycle $\{a, 2-2/a, (a-2)/(a-1), 2/(2-a)\}$ $(a \ne 0, 1, 2)$. So the cluster set consists of these four numbers.

(5) (#18-b) Rewrite the limit as

$$\lim_{n\to\infty} \left[\frac{(2n)!}{n!n^n}\right]^{1/n}$$

and let

$$a_n = \frac{(2n)!}{n!n^n}.$$

Since

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+2)!}{(n+1)!(n+1)^{n+1}}}{\frac{(2n)!}{n!n^n}} = \frac{2(2n+1)}{n+1}\left(\frac{n}{n+1}\right)^n$$

Then since $(1+\frac{1}{n})^n \to e$ as $n \to \infty$, we have

$$\left(\frac{n}{n+1}\right)^n \to e^{-1}.$$

Also, it is easy to see $2(2n+1)/(n+1) \to 4$. Therefore, $\lim_{n\to\infty} a_{n+1}/a_n = 4/e$. By (2.26), we have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 4/e.$$

This gives the answer.