(1) (#6) Let a, b, c, d be rational numbers and x an irrational number such that $cx + d \neq 0$. Prove that $(ax + b)/(cx + d)$ is irrational if and only if $ad \neq bc$.

Proof. We can prove equivalently that $(ax + b)/(cx + d)$ is rational if and only if $ad = bc$.

First, we show $ad = bc \Rightarrow (ax + b)/(cx + d) \in \mathbb{Q}$.

- If $a = 0$ then $bc = 0$. If $b = 0$, we have $(ax + b)/(cx + d) = 0$ is rational; if $c = 0$, since $cx + d \neq 0$, $d \neq 0$ and $(ax + b)/(cx + d) = b/d \in \mathbb{Q}$ since $b, d \in \mathbb{Q}$ and $\mathbb{Q}$ is a field.
- If $a \neq 0$, note that $c \neq 0$ (otherwise, $d = 0$ and $cx + d = 0$ contradicting to the assumption.). Then c/a is rational and nonzero. By $ad = bc$,
$$\frac{ax + b}{cx + d} \cdot \frac{c}{a} = \frac{acx + bc}{acx + ad} = \frac{acx + ad}{acx + ad} = 1,$$
we have
$$\frac{ax + b}{cx + d} = \frac{a}{c},$$
which is again rational.

To show the other direction, suppose $(ax + b)/(cx + d) \in \mathbb{Q}$ is given by m/n for some nonzero integers $m, n \in \mathbb{Z}$. Then we have
$$(am - cn)x = dn - bm,$$
which means either $am - cn = dn - bm = 0$ or $x = (dn - bm)/(am - cn)$. If $am - cn = dn - bm = 0$, then $am = cn$ and $bm = dn$. Then we have the following cases

- First, from $am = cn$, we have $a = 0 \Leftrightarrow c = 0$, in which case $ad = bc = 0$.
- Similarly, from $bm = dn$, we have $b = 0 \Leftrightarrow d = 0$, in which case $ad = bc = 0$.
- At last, none of a, b, c, d is zero. Then $a/c = b/d = n/m$ hence, $ad = bc$.

□

(2) (#10) Prove that for all $n \in \mathbb{N}$ we have
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n - 1}{2n} \leq \frac{1}{\sqrt{3n + 1}}$$
and equality obtains if and only if $n = 1$.

Proof. We prove by induction. When $n = 1$, both sides are $1/2$ hence the inequality and the equality.

When the inequality is true for $n \in \mathbb{N}$, we want to show
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n + 1}{2n + 2} \leq \frac{1}{\sqrt{3n + 4}}.$$
Using the inequality for n, this is to prove
$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \dots \cdot \frac{2n - 1}{2n} \frac{2n + 1}{2n + 2} \leq \frac{1}{\sqrt{3n + 1}} \frac{2n + 1}{2n + 2} \leq \frac{1}{\sqrt{3n + 4}}.$$
So we want to show
$$\frac{2n + 1}{2n + 2} \leq \frac{\sqrt{3n + 1}}{\sqrt{3n + 4}}.$$
Taking a square of both sides and eventually it is equivalent to show
$$(3n + 4)(4n^2 + 4n + 1) \leq (3n + 1)(4n^2 + 8n + 4)$$
that is to show
$$19n \leq 20n.$$

Apparently this is true. So the inequality is proved. Also, since $19n < 20n$ for all $n+1$ case where $n \in \mathbb{N}$, the equality cannot be obtained for $n > 1$. Therefore, the equality is obtained if and only if $n = 1$. □

(3) (#23) If A is an infinite set, then A has a countable infinite subset.

Proof. If A is an infinite set, A is nonempty, choose one element a_1 and let $B_1 = \{a_1\} \subset A$. Assume we have found a subset B_n containing n elements of A for $n \in \mathbb{N}$, note that $A \backslash B_n$ is infinite. (Otherwise, $A = B_n \cup A \backslash B_n$ is finite, a contradiction.) Hence, we can find an element in $A \backslash B_n$, denoted by a_{n+1}, and let $B_{n+1} = B_n \cup \{a_{n+1}\}$. Therefore, B_{n+1} has $n+1$ elements of A. Let

$$B = \bigcup_{n=1}^{\infty} B_n.$$

We have $B = \{x_n : n \in \mathbb{N}\}$ is countable. And for any $n \in \mathbb{N}$, there are n elements $a_1, a_2, \ldots, a_n$ in B. Therefore, B is infinite. □

(4) (#25)

Proof. a). Since every polynomial of degree $n \in \mathbb{N}$ has at most n distinct roots in $\mathbb{C}$, to show that the set of all algebraic numbers is countable, it suffices to show that there are countably many polynomials with integer coefficients. For each $k \in \mathbb{N}$, we consider the number of polynomials

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

such that

$$n + |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0| \leq k.$$

Since a_j and n are integers less and equal than k, there are at most $k+1$ possible values for n $(0, 1, 2, \ldots, k)$ and at most $2k+1$ (including negative integers) possible values of a_j. So there are less than $(k+1)(2k+1)^{k+1}$ (actually much less) polynomials satisfying above condition. Let

$$A_k = \{z \in \mathbb{C} : \sum_{j=0}^{n} a_j z^j = 0, n + \sum_{j=0}^{n} |a_j| \leq k\}$$

for each $k \in \mathbb{N}$. Above shows that A_k is finite hence countable. Then the set of algebraic numbers is $\bigcup_{k=0}^{\infty} A_k$, a countable union of countable sets, hence is countable.

b). This is true because every rational number m/n ($m, n \in \mathbb{Z}$, $n \neq 0$) is a root of $nz - m = 0$, i.e., $a_1 = n$ and $a_0 = -m$. □

(5) Let $A, B \subset \mathbb{R}$, denote

$$A + B = \{a + b : \ a \in A, b \in B\}.$$

Show

$$\inf(A + B) = \inf A + \inf B.$$

Proof.

- If one of A, B is empty, without loss of generality let $A = \emptyset$, then $A + B = \emptyset$. And $\inf \emptyset = \infty$. We have

$$\inf(A + B) = \infty = \inf A + \inf B.$$

- Suppose both A and B are nonempty, $\inf B \neq \infty$. If any of A, B is NOT bounded below, without loss of generality let A unbounded below. Then $\inf A = -\infty$. Pick $b \in B$, for any $n \in \mathbb{N}$, there exists $a_n \in A$ such that $a_n \leq -n - |b|$. This means for any $n \in \mathbb{N}$, there exists an element $a_n + b \in A + B$ such that $a_n + b \leq -n$, which proves $\inf(A + B) = -\infty = \inf A + \inf B$.

- If both A and B are bounded below, $s = \inf A$ and $t = \inf B$ are finite. Then $a \in A$, $b \in B$ implies $s \leq a$ and $t \leq b$. Therefore, $s + t \leq a + b$. Moreover, for any $\varepsilon > 0$, there exists $a \in A$ and $b \in B$ such that $a < s + \varepsilon/2$ and $b < t + \varepsilon/2$. Then $a + b < s + t + \varepsilon$. This proves
$$\inf(A + B) = s + t = \inf A + \inf B.$$

$\square$

(6) (#18) First, since $1 < 2 < 3 < 5$, we have $\frac{1}{5} < \frac{1}{3} < \frac{1}{2} < 1$. Then
$$2^{-(k+1)} = 2^{-k} \cdot \frac{1}{2} < 2^{-k}, 3^{-(k+1)} = 3^{-k} \cdot \frac{1}{3} < 3^{-k}, 5^{-(k+1)} = 5^{-k} \cdot \frac{1}{5} < 5^{-k}$$
for any $k \in \mathbb{N}$, and one has
$$2^{-k} \leq \frac{1}{2}, \quad 3^{-k} \leq \frac{1}{3}, \quad 5^{-k} \leq \frac{1}{5}$$
for any $k \in \mathbb{N}$, which shows $s = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$ is an upper bound of E. Moreover, for any $\varepsilon > 0$, $s \in E$ and $s > s - \varepsilon$. Therefore,
$$\boxed{\sup E = s.}$$

Next, since $5 > 3 > 2 > 0$, by the exponential law (1.24), for any $k, m, n \in \mathbb{N}$, we have 2^{-k}, 3^{-m} and 5^{-n} are positive. Then 0 is a lower bound of E.

<u>Claim:</u> For any $k \in \mathbb{N}$, we have $2^k \geq k$.

We prove the claim by induction. For $k = 1$, $2 > 1$. Assuming $2^k \geq k$, we have
$$2^{k+1} = 2^k \cdot 2 = 2^k + 2^k \geq k + k \geq k + 1.$$
This finishes the proof of the Claim.

Similarly, one also has $3^m \geq m$ and $5^n \geq n$ for $m, n \in \mathbb{N}$.

For any $\varepsilon > 0$, there exists $k, m, n \in \mathbb{N}$ such that
$$k\varepsilon/3 > 1, \quad m\varepsilon/3 > 1, \quad n\varepsilon/3 > 1.$$
This further implies
$$2^{-k} < \varepsilon/3, \quad 3^{-m} < \varepsilon/3, \quad 5^{-n} < \varepsilon/3.$$
Hence, $2^{-k} + 3^{-m} + 5^{-n} < \varepsilon$. This proves
$$\boxed{\inf E = 0.}$$