Math 3150 Final Exam Fall 2015

Name: ____

- The exam will last 2 hours.
- There are 8 problems worth 12 points each.
- One single-sided sheet of theorems and definitions is allowed.
- Use the back sides of the test pages for scratch work, or if you need extra space.

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Total	

Problem 1.

(a) Fix $a > 0$ and consider the power series $f_a(x) = \sum_{n\geq 1} \frac{1}{n}(ax)^n$. Determine its radius of convergence R.

(b) Compute $f_a'(x)$ on $(-R, R)$, and identify this with a known function in closed form. Using the fundamental theorem of calculus, find an explicit expresssion for $f_a(x)$.

(c) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$.

Problem 2. Take for granted that $\sin x$ is differentiable on $\mathbb{R}$, and that $sin'x = \cos x$.

(a) Use the mean value theorem to show that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

(b) Use part (a) to prove that $\sin x$ is uniformly continuous on $\mathbb{R}$.

Problem 3. Let $f, g\colon [a, b] \longrightarrow \mathbb{R}$ be continuous functions such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that there exists some $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

Problem 4. Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is continuous at $x = 0$, but discontinuous everywhere else.

Problem 5. Suppose $f : (S, d) \longrightarrow (S', d')$ is a continuous map between metric spaces.

(a) Give a proof that if $E \subseteq S$ is compact, then $f(E) \subseteq S'$ is compact.

(b) If $E \subseteq S$ is closed, is $f(E) \subseteq S'$ closed? Justify your answer.

Problem 6. Consider the sequences of functions $f_n\colon [-1,1] \longrightarrow \mathbb{R}$ and $g_n\colon [0,1] \longrightarrow \mathbb{R}$ given by $f_n(x) = x^n$ and $g_n(x) = x^n$.

(a) Does either of these sequences converge? If it does, what is its limit? If it doesn't, why not?

(b) Does either of these sequences converge uniformy? Why, or why not?

Problem 7. Let $f\colon [0,3] \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} x & \text{if } x = 1 \text{ or } x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Compute the lower and upper integrals, $L(f)$ and $U(f)$.

(b) Show that f is (Riemann) integrable, and compute $\int_0^3 f(x)dx$.

Problem 8. Let $f\colon [-\pi, \pi] \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \int_0^{x^2} e^{\sin t}\, dt$$

(a) Show that f is differentiable. What is its derivative?

(b) Evaluate $f(0)$ and $f'(\sqrt{\pi/2})$.