Math 3150 Midterm Exam Fall 2014

Problem 1. Complete the following statements.

(a) A sequence x_n in a metric space (M, d) is *Cauchy* if and only if:

Solution. For all $\varepsilon > 0$, there exists an N such that for all $n, m \geq N$, $d(x_m, x_n) < \varepsilon$. □

(b) $a \in \mathbb{R}$ is the *infimum* of a set $A \subset \mathbb{R}$ if and only if:

Solution. $a \leq x$ for all $x \in A$ and for all $\varepsilon > 0$, there exists an $x \in A$ such that $x < a + \varepsilon$. Alternatively, $a \leq x$ for all $x \in A$, and if a' is another point such that $a' \leq x$ for all $x \in A$, then $a' \leq a$. □

(c) A point x is in the *interior* of a set $B \subset \mathbb{R}^n$ if and only if:

Solution. There exists some $\varepsilon > 0$ such that $D(x, \varepsilon) \subset B$. □

(d) A point x is an *accumulation point* of a set $C \subset \mathbb{R}^n$ if and only if:

Solution. For all $\varepsilon > 0$, $D(x, \varepsilon)$ contains some point of C other than x. □

Problem 2. Define a sequence x_n in $\mathbb{R}$ recursively by setting $x_0 = 0$ and $x_n = \sqrt{8 + 2x_{n-1}}$ for $n \geq 1$.

(a) Show by induction that x_n is bounded above by 4.

Solution. $x_0 = 0 \leq 4$ provides the base case. For the inductive case, assume $x_{n-1} \leq 4$. Then

$$x_n = \sqrt{8 + 2x_{n-1}} \leq \sqrt{8 + 2(4)} = \sqrt{16} = 4,$$

completing the induction. □

(b) Show by induction that x_n is monotone increasing. (Hint: multiply and divide $(x_n - x_{n-1})$ by $(x_n + x_{n-1})$.)

Solution. Let $r_n = x_n - x_{n-1}$. We want to show $r_n \geq 0$ for all n. The base case is $r_1 = x_1 - x_0 = \sqrt{8} - 0 \geq 0$. For the inductive case, assume $r_{n-1} \geq 0$. Then

$$\begin{aligned} r_n &= x_n - x_{n-1} \\ &= \frac{(x_n - x_{n-1})(x_n + x_{n-1})}{x_n + x_{n-1}} \\ &= \frac{x_n^2 - x_{n-1}^2}{x_n + x_{n-1}} \\ &= \frac{8 + 2x_{n-1} - 8 - 2x_{n-2}}{x_n + x_{n-1}} \\ &= \frac{2r_{n-1}}{x_n + x_{n-1}} \geq 0, \end{aligned}$$

since $x_n + x_{n-1}$ is positive. □

(c) Prove that x_n converges and compute $\lim_{n\to\infty} x_n$.

Solution. x_n is monotone increasing and bounded above, so by completeness of $\mathbb{R}$, x_n converges to some x. To compute it, we note that

$$x^2 = \lim_{n\to\infty} x_n^2 = 8 + 2 \lim_{n\to\infty} x_{n-1} = 8 + 2x.$$

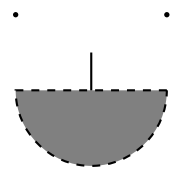
So x is a solution of the quadratic equation $x^2 - 2x - 8 = (x-4)(x+2) = 0$. Either $x = 4$ or $x = -2$, but the latter is ruled out since $x_0 = 0 > -2$ and x_n is increasing. □

Problem 3. Let $A \subset \mathbb{R}^2$ be the set

$$A = \left\{(x_1, x_2) \;\middle|\; x_2 < 0, \text{ and } x_1^2 + x_2^2 < 1\right\} \cup \left\{(0, x_2) \;\middle|\; 0 \le x_2 \le \tfrac{1}{2}\right\} \cup \{(1,1)\} \cup \{(-1,1)\}$$

(a) Draw a picture of A.

Solution.

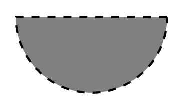


□

(b) What is the interior of A?

Solution. The interior is the set

$$\operatorname{int}(A) = \left\{(x_1, x_2) \;\middle|\; x_2 < 0, \text{ and } x_1^2 + x_2^2 < 1\right\}$$

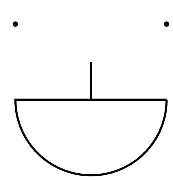


□

(c) What is the boundary of A?

Solution. The boundary is the set

$$\begin{aligned}\operatorname{bd}(A) = &\left\{(x_1, 0) \;\middle|\; -1 \le x_1 \le 1\right\} \\ &\cup \left\{(x_1, x_2) \;\middle|\; -1 \le x_1 \le 1, \; x_2 = -\sqrt{1 - x_1}\right\} \\ &\cup \left\{(0, x_2) \;\middle|\; 0 \le x_2 \le \tfrac{1}{2}\right\} \\ &\cup \{(1,1)\} \cup \{(-1,1)\}.\end{aligned}$$



□

Problem 4. Find the cluster points of the sequence x_n in $\mathbb{R}^2$, where

$$x_n = \left(\sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^n}{2^n}\,,\ \cos\left(\frac{n\pi}{2}\right)\left(1+\frac{1}{n}\right)\right)$$

Solution. Since $\sin(\frac{n\pi}{2})$ and $\cos(\frac{n\pi}{2})$ oscillate between the values 0, 1 and -1 with period 4, while $\frac{(-1)^n}{2^n} \longrightarrow 0$ and $1+\frac{1}{n} \longrightarrow 1$, there are four important subsequences to consider:

$$\begin{aligned}
v_{n(k)} &= x_{4k} = \big(0 + \tfrac{(-1)^{4k}}{2^{4k}}, 1(1+\tfrac{1}{4k})\big) \longrightarrow (0,1)\\
w_{n(k)} &= x_{4k+1} = \big(1 + \tfrac{(-1)^{4k+1}}{2^{4k+1}}, 0(1+\tfrac{1}{4k+1})\big) \longrightarrow (1,0)\\
y_{n(k)} &= x_{4k+2} = \big(0 + \tfrac{(-1)^{4k+2}}{2^{4k+2}}, -1(1+\tfrac{1}{4k+2})\big) \longrightarrow (0,-1)\\
z_{n(k)} &= x_{4k+4} = \big(-1 + \tfrac{(-1)^{4k+3}}{2^{4k+3}}, 0(1+\tfrac{1}{4k+3})\big) \longrightarrow (-1,0)
\end{aligned}$$

Thus the cluster points are $(\pm 1, 0)$ and $(0, \pm 1)$. $\square$

Problem 5. Determine whether the following statements are *true* or *false*. Justify your answers by giving a proof (if true) or a counterexample (if false).

(a) For any set $A \subset \mathbb{R}^n$, if $x \in \mathrm{bd}(A)$ then x is an accumulation point of A.

Solution. **False**. Consider the example

$$A = \{0\}.$$

Then $\mathrm{bd}(A) = A = \{0\}$ but A has no accumulation points. □

(b) For any set A in a metric space M, no point can be simultaneously in $\mathrm{cl}(A)$ and $\mathrm{int}(M \setminus A)$.

Solution. **True**. The closure of A is the complement of the interior of the complement of A:

$$\mathrm{cl}(A) = M \setminus \mathrm{int}(M \setminus A),$$

in particular $\mathrm{cl}(A) \cap \mathrm{int}(M \setminus A) = \emptyset$. □

(c) If $A \subset \mathbb{R}$ has closure $\mathrm{cl}(A) = \mathbb{R}$, then $\mathrm{int}(A) \neq \emptyset$.

Solution. **False**. Consider

$$A = \mathbb{Q} \subset \mathbb{R}.$$

Then $\mathrm{int}(A) = \emptyset$ while $\mathrm{cl}(A) = \mathbb{R}$. □

Problem 6. Let (M, d) be a (not necessarily complete) metric space.

(a) Let p, q, s, and t be any four points in M. Show that

$$d(p,t) \leq d(p,q) + d(q,s) + d(s,t).$$

Solution. Use the triangle inequality twice:

$$\begin{aligned} d(p,t) &\leq d(p,q) + d(q,t) \\ & d(p,q) + d(q,s) + d(s,t).\square \end{aligned}$$

(b) Suppose x_n and y_n are two Cauchy sequences in M, and let $r_n = d(x_n, y_n)$ be the sequence in $\mathbb{R}$ consisting of the distances between their respective terms. Show that r_n converges in $\mathbb{R}$.

(Hint: consider the result of part (a) with $p = x_n$, $q = x_m$, $s = y_m$ and $t = y_n$.)

Solution. From part (a), it follows that

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n).$$

Subtracting $d(x_m, y_m)$ from both sides, we get

$$d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n).$$

Similarly, we can show

$$d(x_m, y_m) - d(x_n, y_n) \leq d(x_n, x_m) + d(y_m, y_n),$$

so that

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m). \tag{1}$$

Since x_n and y_n are Cauchy, given any $\varepsilon > 0$, there exist N_1 and N_2 such that

$$\begin{aligned} n, m \geq N_1 &\implies d(x_n, x_m) < \tfrac{\varepsilon}{2} \\ n, m \geq N_2 &\implies d(y_n, y_m) < \tfrac{\varepsilon}{2}. \end{aligned}$$

Taking $N = \max\{N_1, N_2\}$, it follows from (1) that for all $n, m \geq N$,

$$|r_n - r_m| \leq d(x_n, x_m) + d(y_n, y_m) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Thus r_n is Cauchy, and since $\mathbb{R}$ is complete, r_n converges. $\square$