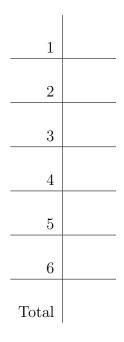
## Math 3150 Midterm Exam Fall 2014

Name: \_\_\_\_\_

- The exam will last 100 minutes.
- There are 6 problems worth 12 points each.
- No notes or other study materials allowed.
- Use the back side of the test pages for scratch work, or if you need extra space.



Problem 1. Complete the following statements.

(a) A sequence  $x_n$  in a metric space (M, d) is Cauchy if and only if:

(b)  $a \in \mathbb{R}$  is the *infimum* of a set  $A \subset \mathbb{R}$  if and only if:

(c) A point x is in the *interior* of a set  $B \subset \mathbb{R}^n$  if and only if:

(d) A point x is an accumulation point of a set  $C\subset \mathbb{R}^n$  if and only if:

**Problem 2.** Define a sequence  $x_n$  in  $\mathbb{R}$  recursively by setting  $x_0 = 0$  and  $x_n = \sqrt{8 + 2x_{n-1}}$  for  $n \ge 1$ .

(a) Show by induction that  $x_n$  is bounded above by 4.

(b) Show by induction that  $x_n$  is monotone increasing. (Hint: multiply and divide  $(x_n - x_{n-1})$  by  $(x_n + x_{n-1})$ .)

(c) Prove that  $x_n$  converges and compute  $\lim_{n\to\infty} x_n$ .

**Problem 3.** Let  $A \subset \mathbb{R}^2$  be the set

 $A = \left\{ (x_1, x_2) \mid x_2 < 0, \text{ and } x_1^2 + x_2^2 < 1 \right\} \cup \left\{ (0, x_2) \mid 0 \le x_2 \le \frac{1}{2} \right\} \cup \left\{ (1, 1) \right\} \cup \left\{ (-1, 1) \right\}$ 

(a) Draw a picture of A.

(b) What is the interior of A?

(c) What is the boundary of A?

**Problem 4.** Find the cluster points of the sequence  $x_n$  in  $\mathbb{R}^2$ , where

$$x_n = \left(\sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^n}{2^n}, \cos\left(\frac{n\pi}{2}\right)\left(1 + \frac{1}{n}\right)\right)$$

**Problem 5.** Determine whether the following statements are *true* or *false*. Justify your answers by giving a proof (if true) or a counterexample (if false).

(a) For any set  $A \subset \mathbb{R}^n$ , if  $x \in bd(A)$  then x is an accumulation point of A.

(b) For any set A in a metric space M, no point can be simultaneously in cl(A) and  $int(M \setminus A)$ .

(c) If  $A \subset \mathbb{R}$  has closure  $cl(A) = \mathbb{R}$ , then  $int(A) \neq \emptyset$ .

**Problem 6.** Let (M, d) be a (not necessarily complete) metric space.

(a) Let p, q, s, and t be any four points in M. Show that

 $d(p,t) \le d(p,q) + d(q,s) + d(s,t).$ 

(b) Suppose  $x_n$  and  $y_n$  are two Cauchy sequences in M, and let  $r_n = d(x_n, y_n)$  be the sequence in  $\mathbb{R}$  consisting of the distances between their respective terms. Show that  $r_n$  converges in  $\mathbb{R}$ .

(Hint: consider the result of part (a) with  $p = x_n$ ,  $q = x_m$ ,  $s = y_m$  and  $t = y_n$ .)