Math 3150 Midterm Exam Fall 2014

Name: ______________________________

- The exam will last 100 minutes.
- There are 6 problems worth 12 points each.
- No notes or other study materials allowed.
- Use the back side of the test pages for scratch work, or if you need extra space.

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Total	

Problem 1. Complete the following statements.

(a) A sequence x_n in a metric space (M, d) is *Cauchy* if and only if:

(b) $a \in \mathbb{R}$ is the *infimum* of a set $A \subset \mathbb{R}$ if and only if:

(c) A point x is in the *interior* of a set $B \subset \mathbb{R}^n$ if and only if:

(d) A point x is an *accumulation point* of a set $C \subset \mathbb{R}^n$ if and only if:

Problem 2. Define a sequence x_n in $\mathbb{R}$ recursively by setting $x_0 = 0$ and $x_n = \sqrt{8 + 2x_{n-1}}$ for $n \geq 1$.

(a) Show by induction that x_n is bounded above by 4.

(b) Show by induction that x_n is monotone increasing. (Hint: multiply and divide $(x_n - x_{n-1})$ by $(x_n + x_{n-1})$.)

(c) Prove that x_n converges and compute $\lim_{n\to\infty} x_n$.

Problem 3. Let $A \subset \mathbb{R}^2$ be the set

$$A = \{(x_1, x_2) \mid x_2 < 0, \text{ and } x_1^2 + x_2^2 < 1\} \cup \{(0, x_2) \mid 0 \leq x_2 \leq \tfrac{1}{2}\} \cup \{(1,1)\} \cup \{(-1,1)\}$$

(a) Draw a picture of A.

(b) What is the interior of A?

(c) What is the boundary of A?

Problem 4. Find the cluster points of the sequence x_n in $\mathbb{R}^2$, where

$$x_n = \left(\sin\left(\frac{n\pi}{2}\right) + \frac{(-1)^n}{2^n}\,,\ \cos\left(\frac{n\pi}{2}\right)\left(1 + \frac{1}{n}\right) \right)$$

Problem 5. Determine whether the following statements are *true* or *false*. Justify your answers by giving a proof (if true) or a counterexample (if false).

(a) For any set $A \subset \mathbb{R}^n$, if $x \in \text{bd}(A)$ then x is an accumulation point of A.

(b) For any set A in a metric space M, no point can be simultaneously in $\text{cl}(A)$ and $\text{int}(M \backslash A)$.

(c) If $A \subset \mathbb{R}$ has closure $\text{cl}(A) = \mathbb{R}$, then $\text{int}(A) \neq \emptyset$.

Problem 6. Let (M, d) be a (not necessarily complete) metric space.

(a) Let p, q, s, and t be any four points in M. Show that

$$d(p,t) \leq d(p,q) + d(q,s) + d(s,t).$$

(b) Suppose x_n and y_n are two Cauchy sequences in M, and let $r_n = d(x_n, y_n)$ be the sequence in $\mathbb{R}$ consisting of the distances between their respective terms. Show that r_n converges in $\mathbb{R}$.

(Hint: consider the result of part (a) with $p = x_n$, $q = x_m$, $s = y_m$ and $t = y_n$.)