MATH 3150 — HOMEWORK 6

Problem 1. Let $f : A \subset (M, d) \longrightarrow \mathbb{R}$ be a uniformly continuous function. Show that f extends uniquely to a continuous function on the closure $\operatorname{cl} A$, i.e., there exists a unique continuous function $\widetilde{f} : \operatorname{cl} A \longrightarrow \mathbb{R}$ such that $\tilde{f} = f$ on A. Here are some hints:

(a) Show that if x_k is a Cauchy sequence in A, then $f(x_k)$ is Cauchy (hence convergent) in $\mathbb{R}$. (Is this true if f is merely continuous in the ordinary sense?)

(b) Show that, if x_k and y_k are sequences in A such that $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x \in \operatorname{cl} A$, then

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f(y_k).$$

(Hint: consider the sequence $x_1, y_1, x_2, y_2, \ldots$)

(c) Use the previous two results to define an extension $\widetilde{f} : \operatorname{cl} A \longrightarrow \mathbb{R}$ of f, and prove that it is continuous and unique.

Solution.

(a) Suppose x_k is Cauchy in A; we want to show that $f(x_k)$ is Cauchy in $\mathbb{R}$. By uniform continuity of f, given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$d(x, x') < \delta \implies \left|f(x) - f(x')\right| < \varepsilon.$$

Then, since x_k is Cauchy, there exists a K such that

$$k, l \geq K \implies d(x_k, x_l) < \delta \implies |f(x_k) - f(x_l)| < \varepsilon.$$

Thus $f(x_k)$ is Cauchy and, by completeness of $\mathbb{R}$, convergent.

(b) Suppose x_k and y_k are sequences in A with the same limit x in $\operatorname{cl} A$. Thus for any $\varepsilon > 0$, there exists K_1 and K_2 such that

$$k \geq K_1 \implies d(x_k, x) < \varepsilon,$$
$$k \geq K_2 \implies d(y_k, x) < \varepsilon.$$

It follows that the interleaved sequence z_k, where $z_{2k+1} = x_k$, $z_{2k} = y_k$, also converges to x, since for $K = 2\max(K_1, K_2)$,

$$k \geq K \implies d(z_k, x) < \varepsilon.$$

Applying part (a) to this sequence, we conclude that $f(z_k)$ converges to some limit $l \in \mathbb{R}$, and since $f(x_k)$ and $f(y_k)$ are subsequences of this sequence, they have the same limit as well.

(c) To define $\widetilde{f}$ for $x \in \operatorname{cl} A$, let x_k be a sequence in A such that $x_k \longrightarrow x$. By part (a), $f(x_k)$ converges, and we set

$$\widetilde{f}(x) = \lim_{k\to\infty} f(x_k).$$

By part (b), this is well-defined (i.e., it does not depend on the choice of sequence converging to x). If $x \in A$ to begin with, it follows by continuity of f that $f(x_k) \longrightarrow f(x)$, so $\widetilde{f}(x) = f(x)$ and $\widetilde{f}$ is therefore an extension of f.

To see that $\widetilde{f}$ is (uniformly) continuous, let $\varepsilon > 0$ be given. We want to show that there exists $\widetilde{\delta} > 0$ such that

$$x, x' \in \operatorname{cl} A, \quad d(x, x') < \widetilde{\delta} \implies \left|\widetilde{f}(x) - \widetilde{f}(x')\right| < \varepsilon.$$

Now, since f is uniformly continuous, there exists $\delta > 0$ such that

$$y, y' \in A, \quad d(y, y') < \delta \implies \left|f(y) - f(y')\right| < \tfrac{\varepsilon}{3}.$$

Let $\widetilde{\delta} = \frac{\delta}{2}$. Then, suppose that $x, x' \in \operatorname{cl} A$ satisfy $d(x, x') < \widetilde{\delta}$. By properties of the closure, there exist $y, y' \in A$ such that

$$d(y, x) < \tfrac{\delta}{4}, \quad d(y', x') < \tfrac{\delta}{4},$$

and by choosing y and y' closer to x and x' if necessary, we can suppose that

$$\left|\widetilde{f}(x) - f(y)\right| < \tfrac{\varepsilon}{3}, \quad \left|\widetilde{f}(x') - f(y')\right| < \tfrac{\varepsilon}{3}.$$

Using the triangle inequality twice, we see

$$d(y, y') \leq d(y, x) + d(x, x') + d(x', y') < \tfrac{\delta}{4} + \tfrac{\delta}{2} + \tfrac{\delta}{4} = \delta.$$

Then

$$\left|\widetilde{f}(x) - \widetilde{f}(x')\right| \leq \left|\widetilde{f}(x) - f(y)\right| + \left|f(y) - f(y')\right| + \left|f(y') - \widetilde{f}(x')\right| < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon.$$

□

Problem 2 (p. 232, #12 (partly))**.** Recall that a map $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called *Lipschitz on A* if there is a constant $L \geq 0$ such that $\|f(x) - f(y)\| \leq L\,\|x - y\|$ for all $x, y \in A$. For the following questions, either provide a proof (for yes) or a counterexample (for no).

(a) Is the sum of two Lipschitz functions again a Lipschitz function?
(b) Is the product of two Lipschitz functions again a Lipschitz function?
(c) Is the sum of two uniformly continuous functions uniformly continuous?
(d) Is the product of two uniformly continuous functions uniformly continuous?

Solution.

(a) Yes. Proof: Suppose $f, g : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are both Lipschitz with constants L_f and L_g, respectively. Then

$$\begin{aligned}\|(f+g)(x) - (f+g)(y)\| &= \|f(x) - f(y) + g(x) - g(y)\| \\ &\leq \|f(x) - f(y)\| + \|g(x) - g(y)\| \leq L_f\,\|x - y\| + L_g\,\|x - y\| = (L_f + L_g)\,\|x - y\|\,.\end{aligned}$$

So $f + g$ is Lipschitz with constant $L = L_f + L_g$.

(b) No. Counterexample: $f(x) = x$ is Lipschitz on all of $\mathbb{R}$ with constant $L = 1$ (obviously). Nevertheless $f(x)f(x) = x^2$ is not, as shown in class (we showed x^2 is not uniformly continuous, and therefore it is not Lipshitz).

If we assume in addition that f and g are both bounded, then fg is Lipschitz:

$$\begin{aligned}\|f(x)g(x) - f(y)g(y)\| &= \|f(x)g(x) - f(x)g(y)\| + \|(f(x)g(y) - f(y)g(y)\| \\ &\leq (\|f(x)\|\,L_g + \|g(y)\|\,L_f)\,\|x - y\|\,.\end{aligned}$$

(c) Yes. Proof: Suppose $f, g : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are uniformly continuous. Given $\varepsilon > 0$, there exist $\delta_f, \delta_g > 0$ such that

$$\begin{aligned}\|x - y\| < \delta_f &\implies \|f(x) - f(y)\| < \varepsilon/2 \\ \|x - y\| < \delta_g &\implies \|g(x) - g(y)\| < \varepsilon/2.\end{aligned}$$

Let $\delta = \min(\delta_f, \delta_g)$. Then $\|x - y\| < \delta$ implies

$$\begin{aligned}\|f(x) + g(x) - (f(y) + g(y))\| &\leq \|f(x) - f(y)\| + \|g(x) - g(y)\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

(d) No. Counterexample: $f(x) = x$ on $\mathbb{R}$. Again this is uniformly continuous (let $\delta = \varepsilon$), but x^2 is not.

If we assume in addition that f and g are bounded, then fg is uniformly continuous. Indeed, given $\varepsilon > 0$ there exist $\delta_f, \delta_g > 0$ such that

$$\|x - y\| < \delta_f \implies \|f(x) - f(y)\| < \frac{\varepsilon}{2(\sup_{z\in A}\|g(z)\| + 1)}$$
$$\|x - y\| < \delta_g \implies \|g(x) - g(y)\| < \frac{\varepsilon}{2(\sup_{z\in A}\|f(z)\| + 1)}.$$

Let $\delta = \min(\delta_f, \delta_g)$. Then $\|x - y\| < \delta$ implies

$$\begin{aligned}\|f(x)g(x) - f(y)g(y)\| &= \|f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)\| \\ &\le \|f(x)\|\,\|g(x) - g(y)\| + \|g(y)\|\,\|f(x) - f(y)\| \\ &< \frac{\varepsilon\,\|f(x)\|}{2(\sup_{z\in A}\|f(z)\| + 1)} + \frac{\varepsilon\,\|g(y)\|}{2(\sup_{z\in A}\|g(z)\| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

□

Problem 3 (p. 235, #37)**.** Prove the following intermediate value theorem for derivatives: If f is differentiable at all points of $[a, b]$ and if $f'(a)$ and $f'(b)$ are non-zero, with opposite signs, then there is a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$. (Note that we do *not* assume that f' is continuous, just that it exists at each $x \in [a, b]$.)

Solution. Since f is differentiable on $[a, b]$, it is continuous there, and since $[a, b]$ is compact, f achieves its maximum and minumum values on $[a, b]$. Either these maximum/minmum values occur in the interior, (a, b), in which case they are local maxima/minima, or they occur at the endpoints. The only way for f to have *no* local max/min in the interior is for it to be strictly increasing or decreasing, but the assumption that $f'(a)$ and $f'(b)$ are nonzero with opposite signs rules this out. Thus there exists at least one local max or min at some $x_0 \in (a, b)$. Finally, we recall that if f has a local maximum or minimum at x_0, then necessarily $f'(x_0) = 0$. □

Problem 4 (p. 235, #38)**.** A real-valued function defined on (a, b) is called *convex* when the following inequality holds for all $x, y \in (a, b)$ and $t \in [0, 1]$:

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y).$$

(In other words, the graph of f between x and y lies on or below the straight line connecting $f(x)$ and $f(y)$.) If f has a continuous second derivative and $f'' > 0$, show that f is convex.

[Hint: Fix $x < y$ and show that the function $g(t) = f(tx + (1 - t)y) - tf(x) - (1 - t)f(y)$ satisfies $g(t) \le 0$ for all $t \in [0, 1]$.]

Solution. Fix x and y in (a, b) with $x < y$ and consider the function $g : [0, 1] \longrightarrow \mathbb{R}$ defined by

$$g(t) = f(tx + (1 - t)y) - tf(x) - (1 - t)f(y).$$

We want to show that $g(t) \le 0$ for all $t \in [0, 1]$. Now g is twice differentiable on $(0, 1)$, with

$$g''(t) = f''(tx + (1 - t)y)(x - y)^2.$$

Since $x < y$ and $f'' > 0$ by assumption, it follows that $g'' > 0$ for all $t \in (0, 1)$.

Continuity of f on $[x, y]$ implies continuity of g on $[0, 1]$, so g must attain its maximum and minimum values there, but since $g'' > 0$, any local extrema in $(0, 1)$ must be minima. It follows that the maxima of g occur at $t = 0$ and/or $t = 1$. But we note that

$$g(0) = g(1) = 0$$

so therefore $g(t) \leq 0$ for all $t \in [0,1]$. □

Problem 5 (p. 336, #44)**.** Let $f : [0,1] \longrightarrow \mathbb{R}$ be Riemann integrable and suppose that for every a, b with $0 \leq a < b \leq 1$ there exists a c with $a < c < b$ with $f(c) = 0$. Prove that $\int_0^1 f\,dx = 0$. Must f be zero? What if f is continuous?

Solution. By the assumption that f is Riemann integrable,

$$\int_0^1 f(x)\,dx = \sup_P \{L(f,[0,1],P)\} = \inf_P \{U(f,[0,1],P)\}$$

exists, so for every $\varepsilon > 0$ there exist partitions P_0 and P_1 of $[0,1]$ such that

$$U(f,[0,1],P_0) < \int_0^1 f(x)\,dx - \varepsilon, \quad \text{and} \quad L(f,[0,1],P_1) > \int_0^1 f(x)\,dx + \varepsilon.$$

Passing to the common refinement $P = P_0 \cup P_1$ if necessary (which only decreases U and increases L), we may assume that $P = P_0 = P_1$..

On each interval $[x_{i-1}, x_i]$ of P, there exists a point c where $f(c) = 0$, so it must be that

$$\sup_{[x_{i-1},x_i]} f(x) \geq 0, \quad \text{and} \quad \inf_{[x_{i-1},x_i]} f(x) \leq 0,$$

and putting these estimates into the upper and lower sums gives

$$U(f,[0,1],P) \geq 0, \quad L(f,[0,1],P) \leq 0.$$

Thus we have shown that for every $\varepsilon > 0$, there exists a P such that

$$0 \leq U(f,[0,1],P) < \int_0^1 f(x)\,dx - \varepsilon, \quad \text{and}$$
$$0 \geq L(f,[0,1],P) > \int_0^1 f(x)\,dx + \varepsilon.$$

which implies that $\int_0^1 f(x)\,dx = 0$.

f need not vanish identically, as the example

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$$

shows. However if f is assumed to be continuous in addition, then we must have $f(x) = 0$ for all x. □