MATH 3150 — HOMEWORK 6

Problem 1. Let $f : A \subset (M, d) \longrightarrow \mathbb{R}$ be a uniformly continuous function. Show that f extends uniquely to a continuous function on the closure $\mathrm{cl}\, A$, i.e., there exists a unique continuous function $\widetilde{f} : \mathrm{cl}\, A \longrightarrow \mathbb{R}$ such that $\tilde{f} = f$ on A. Here are some hints:

(a) Show that if x_k is a Cauchy sequence in A, then $f(x_k)$ is Cauchy (hence convergent) in $\mathbb{R}$. (Is this true if f is merely continuous in the ordinary sense?)

(b) Show that, if x_k and y_k are sequences in A such that $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = x \in \mathrm{cl}\, A$, then

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f(y_k).$$

(Hint: consider the sequence $x_1, y_1, x_2, y_2, \ldots$)

(c) Use the previous two results to define an extension $\widetilde{f} : \mathrm{cl}\, A \longrightarrow \mathbb{R}$ of f, and prove that it is continuous and unique.

Problem 2 (p. 232, #12 (partly))**.** Recall that a map $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called *Lipschitz on A* if there is a constant $L \geq 0$ such that $\|f(x) - f(y)\| \leq L\,\|x - y\|$ for all $x, y \in A$. For the following questions, either provide a proof (for yes) or a counterexample (for no).

(a) Is the sum of two Lipschitz functions again a Lipschitz function?

(b) Is the product of two Lipschitz functions again a Lipschitz function?

(c) Is the sum of two uniformly continuous functions uniformly continuous?

(d) Is the product of two uniformly continuous functions uniformly continuous?

Problem 3 (p. 235, #37)**.** Prove the following intermediate value theorem for derivatives: If f is differentiable at all points of $[a, b]$ and if $f'(a)$ and $f'(b)$ are non-zero, with opposite signs, then there is a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$. (Note that we do *not* assume that f' is continuous, just that it exists at each $x \in [a, b]$.)

Problem 4 (p. 235, #38)**.** A real-valued function defined on (a, b) is called *convex* when the following inequality holds for all $x, y \in (a, b)$ and $t \in [0, 1]$:

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

If f has a continuous second derivative and $f'' > 0$, show that f is convex.

[Hint: Fix $x < y$ and show that the function $g(t) = f(tx + (1 - t)y) - tf(x)$ -- $(1 - t)f(y)$ satisfies $g(t) \leq 0$ for all $t \in [0, 1]$.]

Problem 5 (p. 336, #44)**.** Let $f : [0, 1] \longrightarrow \mathbb{R}$ be Riemann integrable and suppose that for every a, b with $0 \leq a < b \leq 1$ there exists a c with $a < c < b$ with $f(c) = 0$. Prove that $\int_0^1 f\, dx = 0$. Must f be zero? What if f is continuous?