MATH 3150 — HOMEWORK 5

Problem 1 (p. 172, #1)**.** Which of the following sets are connected? Which are compact?
(a) $\{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| \leq 1\}$
(b) $\{x \in \mathbb{R}^n \mid \|x\| \leq 10\}$
(c) $\{x \in \mathbb{R}^n \mid 1 \leq \|x\| \leq 2\}$
(d) $\mathbb{Z} = \{\text{integers in } \mathbb{R}\}$
(e) a finite set in $\mathbb{R}$
(f) $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ (Be careful with the case $n = 1$!)
(g) Boundary of the unit square in $\mathbb{R}^2$
(h) The boundary of a bounded set in $\mathbb{R}$
(i) The rationals in $[0, 1]$
(j) A closed set in $[0, 1]$

Solution.
(a) Connected, noncompact.
(b) Connected, compact.
(c) Compact. Connected if $n \geq 2$, not connected if $n = 1$.
(d) Not connected, not compact.
(e) Connected if just one point, otherwise not connected. Is compact.
(f) Compact. Connected if $n \geq 2$, not connected if $n = 1$, where the set is $\{\pm 1\} \subset \mathbb{R}$.
(g) Connected, compact.
(h) Always compact (the boundary of a set A is always closed, being the intersection of closed sets $\mathrm{cl}(A)$ and $\mathrm{cl}(\mathbb{R} \setminus A)$, and is bounded if the set is bounded). May or may not be connected: for example $\mathrm{bd}([0, 1]) = \{0, 1\}$ is not connected, while $\mathrm{bd}(\{0\}) = \{0\}$ is connected.
(i) Neither connected nor compact.
(j) Compact; may or may not be connected.

□

Problem 2 (p. 191, #4)**.** Let $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous, $x, y \in A$ and $c : [0, 1] \longrightarrow A \subset \mathbb{R}^n$ be a continuous curve joining x and y. Show that along this curve, f attains its maximum and minimum values (among all values along the curve).

Solution. Since composition of continuous functions is continuous, $f \circ c : [0, 1] \longrightarrow \mathbb{R}$ is continuous. The domain $[0, 1]$ is compact, so $f \circ c$ attains its maximum and minimum values (owing to compactness of $(f \circ c)([0, 1])$). This is the same as the statement to be shown. □

Problem 3 (p. 193, #3)**.** Let $f : [0, 1] \longrightarrow [0, 1]$ be continuous. Prove that f has a fixed point (i.e. a point $x \in [0, 1]$ such that $f(x) = x$).

Solution. Since f is continuous, $g(x) = f(x) - x$ is continuous. A fixed point is the same thing as a point $x_0 \in [0, 1]$ where $g(x_0) = 0$.

Suppose there are no fixed points. Since $g([0, 1])$ is connected, it must be that either $g(x) > 0$ for all $x \in [0, 1]$ or $g(x) < 0$ for all $x \in [0, 1]$. If $g(x) > 0$, then $f(x) > x$ for all $x \in [0, 1]$, but then $f(1) > 1$ which contradicts the assumption on the range of f: that $f : [0, 1] \longrightarrow [0, 1]$. On the other hand, if $g(x) < 0$, then $f(x) < x$ for all $x \in [0, 1]$, but then $f(0) < 0$ which also contradicts the assumption on the range. Thus there must be some x such that $g(x) = 0$, or equivalently $f(x) = x$.

Alternatively, we can note that $g(0) = f(0) \in [0,1]$ and $g(1) = f(1) - 1 \in [-1,0]$, and by the intermediate value theorem, for any $c \in [g(1), g(0)]$, there exists x_0 such that $g(x_0) = c$. In particular, $c = 0$ always lies in $[g(1), g(0)]$, so there exists a fixed point. □

Problem 4 (p. 174, #21)**.**

(a) Prove that a set $A \subset (M, d)$ is connected if and only if $\emptyset$ and A are the only subsets of A that are open and closed relative to A. (A set $U \subset A$ is called *open relative to* A if $U = V \cap A$ for some open set $V \subset M$; 'closed relative to A' is defined similarly.)

(b) Prove that $\emptyset$ and $\mathbb{R}^n$ are the only subsets of $\mathbb{R}^n$ that are both open and closed.

Proof.

(a) A is *not* connected if and only if there exist separating open sets $U, V \subset M$ such that
   (1) $A = (A \cap U) \cup (A \cap V)$,
   (2) $A \cap U \neq \emptyset$,
   (3) $A \cap V \neq \emptyset$,
   (4) $(A \cap U) \cap (A \cap V) = \emptyset$.
   Equivalently, $U' = A \cap U$ and $V' = A \cap V$ are nonempty, relatively open sets such that $U' = A \setminus V'$ and $V' = A \setminus U'$; in turn, this holds if and only if U' is a nonempty open set in A which is not all of A and which is both open and closed. Since all the implications are if and only if, the proof is complete.

(b) $\mathbb{R}^n$ is path-connected, since any points $x, y \in \mathbb{R}^n$ are connected by the path $c(t) = (1-t)x + ty$, hence is is connected. By part (a), it follows that the only subsets if it which are open and closed are $\emptyset$ and $\mathbb{R}^n$.

□

Problem 5. Let (M_1, d_1) and (M_2, d_2) be metric spaces with compact sets $K_1 \subset M_1$ and $K_2 \subset M_2$. Show that $K_1 \times K_2$ is a compact subset of the space $(M_1 \times M_2, d = d_1 + d_2)$. (The metric d on the product $M_1 \times M_2$ is defined by $d\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2)$.)

Solution. By Bolzano-Weierstrass, we may replace 'compact' by 'sequentially compact'. Let (x_n, y_n) be a sequence in $K_1 \times K_2$. We are done if we show that it has a subsequence which converges in $K_1 \times K_2$.

Since K_1 is sequentially compact, there is a subsequence $x_{n(k)}$ which converges in K_1:

$$x_{n(k)} \overset{k\to\infty}{\longrightarrow} x \in K_1.$$

Then consider the sequence $y_{n(k)}$, $k \in \mathbb{N}$, in K_2. Since K_2 is sequentially compact, this has a further subsequence $y_{n(k(l))}$, $l \in \mathbb{N}$ which converges in K_2:

$$y_{n(k(l))} \overset{l\to\infty}{\longrightarrow} y \in K_1.$$

The subsequence $x_{n(k(l))}$ of $x_{n(k)}$ also converges to x (since a subsequence of a convergent sequence always converges to the same limit), thus

$$(x_{n(k(l))}, y_{n(k(l))}) \longrightarrow (x, y) \in K_1 \times K_2$$

is a convergent subsequence of the original. □