MATH 3150 — HOMEWORK 4

Problem 1 (p. 125, #2)**.** Let (M, d) be a metric space with the property that every bounded sequence has a convergent subsequence. Prove that M is complete.

Solution. We want to show that every Cauchy sequence in M converges. Suppose x_n is a Cauchy sequence in M. Then x_n is bounded, as proved in class, so by assumption it has a convergent subsequence. But if a subsequence of a Cauchy sequence converges to x, then the Cauchy sequence converges to x, so we conclude that x_n converges. $\square$

Problem 2 (p. 231, #1)**.**

(a) Prove directly (i.e. with 'ε's and 'δ's) that the function $1/x^2$ is continuous on $(0, \infty)$.

(b) A constant function $f : A \longrightarrow \mathbb{R}^m$ is a function such that $f(x) = f(y)$ for all $x, y \in A$. Show that f is continuous.

(c) Is the function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(y) = 1/(y^4 + y^2 + 1)$ continuous? Justify your answer.

Solution.

(a) Let $x_0 \in (0, \infty)$. Given $\varepsilon > 0$, we have to find a $\delta > 0$ such that

$$|x - x_0| < \delta \implies \left|1/x^2 - 1/x_0^2\right| < \varepsilon.$$

To estimate the latter, we notice

$$\left|\frac{1}{x^2} - \frac{1}{x_0^2}\right| = \frac{\left|x_0^2 - x^2\right|}{x_0^2 x^2} = \frac{|x + x_0|\,|x - x_0|}{x_0^2 x^2}. \tag{1}$$

We are not allowed to let δ depend on x, though it may depend on ε and x_0. The term $|x - x_0|$ is good, since this will be less than δ, but we have to find a way to estimate the other factors. Notice that if we require

$$|x - x_0| < x_0/2,$$

then we can conclude

$$|x| > x_0/2, \quad \text{and} \quad |x + x_0| < x_0 + 3x_0/2 = 5x_0/2.$$

It then follows that (1) is estimated by

$$\frac{|x + x_0|\,|x - x_0|}{x_0^2 x^2} < \frac{5x_0}{2}\frac{1}{x_0^2\,(x_0/2)^2}\,|x - x_0| = \frac{10}{x_0^3}\,|x - x_0|\,.$$

Thus, given $\varepsilon > 0$, we choose $\delta = \min\left\{x_0/2, 10\varepsilon/x_0^3\right\}$. Then if $|x - x_0| < \delta$, it follows by the above computations that

$$\left|\frac{1}{x^2} - \frac{1}{x_0^2}\right| = \frac{|x + x_0|\,|x - x_0|}{x_0^2 x^2} < \frac{10}{x_0^3}\,|x - x_0| \le \varepsilon.$$

(b) Let f be the constant function $f(x) = c \in \mathbb{R}^m$ for all $x \in A$. Let $B \subset \mathbb{R}^m$ be a closed set. If $c \in B$, then $f^{-1}(B) = A$, which is closed in A, and if $c \notin B$, then $f^{-1}(B) = \emptyset$, which is also closed in A. Thus f is continuous.

Alternatively, given any $\varepsilon > 0$, we can choose $\delta > 0$ however we like, say $\delta = 1$. Then for $d(x, y) < \delta$,

$$|f(x) - f(y)| = |c - c| = 0 < \varepsilon.$$

(c) The function is continuous. Indeed, $g(y) = 1 + y^2 + y^4$ is a sum of products of continuous functions, hence continuous. Also, $g(y) \neq 0$ for any y since $y^2 \geq 0$ and $y^4 \geq 0$ so $g(y) \geq 1$. Then $f(y) = 1/g(y)$ is also continuous.

$\square$

Problem 3. Define maps $s : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $m : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as addition and scalar multiplication:

$$s(x, y) = x + y, \quad \text{and} \quad m(\lambda, x) = \lambda x.$$

Show that these maps are continuous.

Solution. We use the sequential characteriztaion of continuity. Thus consider a convergent sequence $(x_n, y_n) \longrightarrow (x, y)$ in the domain $\mathbb{R}^n \times \mathbb{R}^n$ for s. Since $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and convergence in $\mathbb{R}^n$ is equivalent to convergence of component sequences, it follows that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in $\mathbb{R}^n$. By the limit theorem for $\mathbb{R}^n$, the sequence

$$s(x_n, y_n) = x_n + y_n \longrightarrow x + y = s(x, y).$$

This implies that s is continuous.

Likewise, suppose $(\lambda_n, x_n) \longrightarrow (\lambda, x)$ is a convergent sequence in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$. In particular $\lambda_n \longrightarrow \lambda$ in $\mathbb{R}$ and $x_n \longrightarrow x$ in $\mathbb{R}^n$. Then

$$m(\lambda_n, x_n) = \lambda_n \, x_n \longrightarrow \lambda \, x = m(\lambda, x),$$

so m is continuous. $\square$

Problem 4 (p. 182, #5, p. 184, #3)**.**

(a) Give an example of a continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}$ and an open subset $A \subset \mathbb{R}$ such that $f(A)$ is *not* open.

(b) Give an example of a continuous map $f : \mathbb{R} \longrightarrow \mathbb{R}$ and a closed subset $B \subset \mathbb{R}$ such that $f(B)$ is *not* closed.

Solution. (a) Take, for instance, $f(x) = 0$ for all x, and A to be any nonempty open set. Then $F(A) = \{0\}$, which is not open.

(b) Consider

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = x^2/(x^2 + 1),$$
$$B = [0, \infty) \subset \mathbb{R}.$$

The function is continuous since the map $x \longmapsto x^2$ is continuous and the map $x \longmapsto x^2 + 1$ are continuous and nonvanishing. However $f(B) = [0, 1) \subset \mathbb{R}$, which is not closed.

$\square$

Problem 5 (p. 232, #9)**.** Prove the following "gluing lemma": *Let $f : [a, b] \longrightarrow \mathbb{R}^m$ and $g : [b, c] \longrightarrow \mathbb{R}^m$ be continuous, and such that $f(b) = g(b)$. Define $h : [a, c] \longrightarrow \mathbb{R}^m$ by $h = f$ on $[a, b]$ and $h = g$ on $[b, c]$. Then h is continuous.* Generalize this result to sets $A, B \subset (M, d)$ in a metric space, with functions $f : A \longrightarrow (N, \rho)$ and $g : B \longrightarrow (N, \rho)$.

Solution. The general statement is that if $A, B \subset (M, d)$ are closed sets, and $f : A \longrightarrow (N, \rho)$ and $g : B \longrightarrow (N, \rho)$ are continuous functions such that $f = g$ on $A \cap B$, then there exists a continuous function

$$h : A \cup B \longrightarrow (N, \rho), \quad \text{such that}$$
$$h(x) = f(x), \text{ if } x \in A, \quad h(x) = g(x), \text{ if } x \in B.$$

We *define* h by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$; the hypothesis that $f = g$ on $A \cap B$ means that this is well-defined.

To see that h is continuous, suppose $x_k \longrightarrow x$ in $A \cup B$. We deal with three cases. First if $x \in A$ but $x \notin A \cap B$, then $x \in A \setminus (A \cap B)$, which is an open set relative to A since $A \cap B$ is closed. This implies that $x_k \in A$ for k sufficiently large (i.e. there exists an N such that $x_k \in A$ for $k \geq N$) and then $f(x_k) \longrightarrow f(x)$ which means that $h(x_k) \longrightarrow h(x)$. The case that $x \in B$ but $x \notin A \cap B$ is similar.

Finally, suppose that $x \in A \cap B$. Given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$x', x \in A,\ d(x', x) < \delta_1 \implies \rho(f(x'), f(x)) < \varepsilon,$$
$$x'', x \in B,\ d(x'', x) < \delta_2 \implies \rho(g(x''), g(x)) < \varepsilon.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$ and taking $N \in \mathbb{N}$ so that $d(x_k, x) < \delta$ for all $k \geq N$, it follows that for $k \geq N$,

$$|h(x_k) - h(x)| = \begin{cases} |f(x_k) - f(x)| < \varepsilon, & x_k \in A, \\ |g(x_k) - g(x)| < \varepsilon, & x_k \in B, \end{cases}$$

and thus that $h(x_k) \longrightarrow h(x)$. Since x_k was an arbitrary convergent sequence in $A \cup B$, we conclude that h is continuous.

Alternatively, we can argue as follows. With h defined as above, consider a closed set $C \subset h(A \cup B)$. $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ as can be seen from the definition of h. Now $f^{-1}(C)$ and $g^{-1}(C)$ are relatively closed in A and B, respectively, since f and g are continuous. Since A and B are themselves closed, $(f^{-1}(C) \cap A) \subset A \cup B$ is closed relative to $A \cup B$ and likewise for $g^{-1}(C) \cap B) \subset A \cup B$. Thus $h^{-1}(C)$ is closed and h is therefore continuous.

There is a version of the theorem also for A and B both open, with $f = g$ on $A \cap B$. In this case we consider $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$ for an open set $U \subset N$, and deduce that it is relatively open. $\square$