MATH 3150 — HOMEWORK 3

Problem 1 (p. 70, #1, #3, #4)**.** This problem concerns the vector space $C([0,1])$ of continuous, real-valued functions $f:[0,1]\longrightarrow\mathbb{R}$, equipped with the inner product and two different norms:

$$\langle f,g\rangle=\int_0^1 f(x)g(x)\,dx,\quad \|f\|_2=\sqrt{\langle f,f\rangle},\quad \|f\|_\infty=\sup\{|f(x)|:x\in[0,1]\}.$$

(a) For $f(x)=1$ and $g(x)=x$, find $d(f,g)$ for both the sup norm $\|\cdot\|_\infty$ and the L^2-norm $\|\cdot\|_2$.
(b) Verify the Cauchy-Schwarz inequality for $f(x)=1$ and $g(x)=x$.
(c) Verify the triangle inequality for $f(x)=x$ and $g(x)=x^2$ in both norms.

Solution.
(a)

$$d_\infty(1,x)=\|1-x\|_\infty=\sup\{|1-x|:x\in[0,1]\}=1,$$
$$d_2(1,x)=\|1-x\|_2=\left(\int_0^1(1-x)^2\,dx\right)^{1/2}=\tfrac{1}{\sqrt{3}}.$$

(b) On the one hand we have

$$|\langle 1,x\rangle|=\left|\int_0^1 1\cdot x\,dx\right|=\tfrac{1}{2}.$$

On the other hand,

$$\|1\|_2=\left(\int_0^1 1^2\,dx\right)^{1/2}=1,\quad\text{and}$$
$$\|x\|_2=\left(\int_0^1 x^2\,dx\right)^{1/2}=\tfrac{1}{\sqrt{2}}.$$

Since $2>\sqrt{2}$, we have

$$|\langle 1,x\rangle|=\tfrac{1}{2}<\tfrac{1}{\sqrt{2}}=\|1\|_2\,\|x\|_2\,.$$

(c) In the sup norm,

$$\left\|x+x^2\right\|_\infty=\sup\left\{x+x^2:x\in[0,1]\right\}=2,$$
$$\|x\|_\infty=\sup\{x:x\in[0,1]\}=1,$$
$$\left\|x^2\right\|_\infty=\sup\{x:x\in[0,1]\}=1.$$

So indeed $\left\|x+x^2\right\|_\infty=2\le 2=\|x\|_\infty+\left\|x^2\right\|_\infty$. For the L^2 norm,

$$\left\|x+x^2\right\|_2=\left(\int_0^1(x+x^2)^2\,dx\right)^{1/2}=\tfrac{\sqrt{31}}{\sqrt{30}},$$
$$\|x\|_2=\left(\int_0^1 x^2\,dx\right)^{1/2}=\tfrac{1}{\sqrt{2}},$$
$$\left\|x^2\right\|_2=\left(\int_0^1 x^4\,dx\right)^{1/2}=\tfrac{1}{\sqrt{5}},$$

and indeed,

$$\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}=\frac{\sqrt{5}+\sqrt{2}}{\sqrt{10}}=\frac{\sqrt{15}+\sqrt{6}}{\sqrt{30}}\ge\frac{\sqrt{31}}{\sqrt{30}}.$$

□

Problem 2 (p. 108, #4)**.** Let $B\subset\mathbb{R}^n$ be any set. Define

$$C=\{x\in\mathbb{R}^n:d(x,y)<1\text{ for some }y\in B\}.$$

Show that C is open.

Solution. Let $x \in C$; we must show that there exists an $\varepsilon > 0$ such that $D(x, \varepsilon) \subset C$. By definition, there exists some $y \in B$ such that $d(x, y) < 1$, and we set $\varepsilon = 1 - d(x, y)$. Then

$$\begin{aligned} z \in D(x,\varepsilon) &\iff d(z,x) < \varepsilon \\ &\implies d(z,y) \le d(z,x) + d(x,y) < \varepsilon + d(x,y) = 1 \end{aligned}$$

so $D(x, \varepsilon) \subset C$. □

Problem 3 (p. 108, #6)**.**

(a) In $\mathbb{R}^2$, show that

$$\|x\| \le \|x\|_1 \le \sqrt{2}\,\|x\|$$

where $\|x\|_1 = |x_1| + |x_2|$ is the taxicab norm, and $\|x\| = \sqrt{x_1^2 + x_2^2}$ is the usual Euclidean norm.

(b) Use the results of the first part to show that $\mathbb{R}^2$ with the taxicab metric $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$ has the same open sets as it does with the standard metric $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. In other words, show that every set which is open with respect to d is also open with respect to d_1 and vice versa.

Solution. (a) Consider

$$\|x\|^2 = |x_1|^2 + |x_2|^2 \le |x_1|^2 + |x_2|^2 + 2\,|x_1|\,|x_2| = (|x_1| + |x_2|)^2 = \|x\|_1^2 .$$

Taking square roots of both sides gives $\|x\| \le \|x\|_1$. For the other inequality, first note that for any positive numbers a and b,

$$0 \le (a-b)^2 = a^2 + b^2 - 2ab \implies 2ab \le a^2 + b^2.$$

Thus,

$$\begin{aligned} \|x\|_1^2 &= (|x_1| + |x_2|)^2 \\ &= |x_1|^2 + |x_2|^2 + 2\,|x_2|\,|x_2| \\ &\le |x_1|^2 + |x_2|^2 + |x_1|^2 + |x_2|^2 \\ &= 2\left(|x_1|^2 + |x_2|^2\right) = 2\,\|x\|^2 . \end{aligned}$$

Again, taking square roots of both sides shows $\|x\|_1 \le \sqrt{2}\,\|x\|$.

(b) Suppose $U_2 \subset \mathbb{R}^2$ is open with respect to $d_2 = d$. If $x \in U_2$, then by definition there exists $\varepsilon > 0$ such that

$$D_2(x,\varepsilon) = \left\{ y \;\middle|\; \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} < \varepsilon \right\} \subset U_2.$$

To show that U_2 is open with respect to d_1 we must produce an $\varepsilon' > 0$ such that $D_1(x, \varepsilon') \subset U_2$, and it is sufficient to produce one such that $D_1(x, \varepsilon') \subset D_2(x, \varepsilon) \subset U_2$. From part (a) it follows that

$$\begin{aligned} y \in D_1(x,\varepsilon) &\iff d_1(x,y) < \varepsilon \\ &\implies d_2(x,y) \le d_1(x,y) < \varepsilon \iff y \in D_2(x,\varepsilon). \end{aligned}$$

Thus $D_1(x, \varepsilon') \subset D_2(x, \varepsilon)$ for $\varepsilon' = \varepsilon$.

In the other direction, suppose $U_1 \subset \mathbb{R}^2$ is open with respect to d_1. Given an arbitrary $x \in U_1$, we must find ε' such that $D_2(x, \varepsilon') \subset U_1$ and it suffices to choose ε' such that $D(x, \varepsilon') \subset D_1(x, \varepsilon)$.

The other direction is similar. To show that a d_1 open set U_1 is also open with respect to d_2, it suffices to show that, given $D_1(x, \varepsilon)$ for some ε, we can produce an $\varepsilon' > 0$ such that $D_2(x, \varepsilon') \subset D_1(x, \varepsilon)$. From part (a) it follows that

$$y \in D_2(x, \tfrac{1}{\sqrt{2}}\varepsilon) \iff d_2(x,y) < \tfrac{1}{\sqrt{2}}\varepsilon \implies d_1(x,y) < \sqrt{2}\tfrac{1}{\sqrt{2}}\varepsilon = \varepsilon \iff y \in D_1(x,\varepsilon)$$

or in other words, $D_2(x, \varepsilon') \subset D_1(x, \varepsilon)$ for $\varepsilon' = \frac{1}{\sqrt{2}}\varepsilon$.

□

Problem 4 (p. 145, #12, #14)**.** Prove the following properties for subsets A and B of a metric space:

(a) $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$.
(b) $\operatorname{int}(A \cup B) \supset \operatorname{int}(A) \cup \operatorname{int}(B)$.
(c) $\operatorname{int}(A \cap B) = \operatorname{int}(A) \cap \operatorname{int}(B)$.
(d) $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$.
(e) $\operatorname{cl}(A \cap B) \subset \operatorname{cl}(A) \cap \operatorname{cl}(B)$.
(f) $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$.

Solution.

(a) The interior of an open set is the open set itself, and $\operatorname{int}(A)$ is open.

(b) Suppose $x \in \operatorname{int}(A) \cup \operatorname{int}(B)$. Thus x lies in one and/or the other; without loss of generality assume $x \in \operatorname{int}(A)$. Then there exists $\varepsilon > 0$ such that $D(x, \varepsilon) \subset A$, and then

$$D(x, \varepsilon) \subset A \cup B \implies x \in \operatorname{int}(A \cup B).$$

(c) Suppose $x \in \operatorname{int}(A) \cap \operatorname{int}(B)$, so that $x \in \operatorname{int}(A)$ and $x \in \operatorname{int}(B)$. Then there exist $\varepsilon, \varepsilon' > 0$ such that

$$\begin{aligned} D(x, \varepsilon) \subset A, \quad D(x, \varepsilon') \subset B, \\ \implies D(x, \min\{\varepsilon, \varepsilon'\}) \subset A \cap B, \end{aligned}$$

which means $x \in \operatorname{int}(A \cap B)$.

In the other direction, suppose $x \in \operatorname{int}(A \cap B)$. Then there exists $\varepsilon > 0$ such that $D(x, \varepsilon) \subset A \cap B$, thus

$$D(x, \varepsilon) \subset A, \quad \text{and} \quad D(x, \varepsilon) \subset B$$

which means that $x \in \operatorname{int}(A) \cap \operatorname{int}(B)$.

(d) The remaining three follow from the first three, along with the fact that complements of interiors are unions and vice versa, and the complement of the closure is the interior of the complement. Thus

$$\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(M \setminus \operatorname{int}(M \setminus A)) = M \setminus \operatorname{int}(\operatorname{int}(M \setminus A) = M \setminus \operatorname{int}(M \setminus A) = \operatorname{cl}(A).$$

(e)

$$\begin{aligned} \operatorname{cl}(A \cap B) &= M \setminus \operatorname{int}(M \setminus (A \cap B)) = M \setminus \operatorname{int}((M \setminus A) \cup (M \setminus B)) \\ &\subset M \setminus (\operatorname{int}(M \setminus A) \cup \operatorname{int}(M \setminus B)) = (M \setminus \operatorname{int}(M \setminus A)) \cap (M \setminus \operatorname{int}(M \setminus B)) \\ &= \operatorname{cl}(A) \cap \operatorname{cl}(B). \end{aligned}$$

(f)

$$\begin{aligned} \operatorname{cl}(A \cup B) &= M \setminus \operatorname{int}(M \setminus (A \cup B)) = M \setminus \operatorname{int}((M \setminus A) \cap (M \setminus B)) \\ &= M \setminus (\operatorname{int}(M \setminus A) \cap \operatorname{int}(M \setminus B)) = (M \setminus \operatorname{int}(M \setminus A)) \cap (M \setminus \operatorname{int}(M \setminus B)) \\ &= \operatorname{cl}(A) \cup \operatorname{cl}(B). \end{aligned}$$

□

Problem 5 (p. 143, #1, #2)**.** Determine whether the following sets are open or closed, and each set find its interior, closure and boundary.

(a) $(1, 2)$ in $\mathbb{R}^1 = \mathbb{R}$
(b) $[2, 3]$ in $\mathbb{R}$
(c) $\bigcap_{n=1}^{\infty}[-1, 1/n)$ in $\mathbb{R}$
(d) $\mathbb{R}^n$ in $\mathbb{R}^n$
(e) $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$

(f) $\{r \in (0,1) \mid r \text{ is rational}\}$ in $\mathbb{R}$
(g) $\{(x,y) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$ in $\mathbb{R}^2$
(h) $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$ in $\mathbb{R}^n$
(i) $\{x_k \in \mathbb{R}^n\}$ for a sequence x_k in $\mathbb{R}^n$ with no repeated terms.

Proof. (a) Open, interior is $(1,2)$, closure is $[1,2]$, boundary is $\{1,2\}$.
(b) Closed, interior is $(2,3)$, closure is $[2,3]$, boundary is $\{2,3\}$.
(c) The set itself is $[-1,0]$, so closed, with interior $(-1,0)$, closure $[-1,0]$ and boundary $\{-1,0\}$.
(d) Open and closed, interior = closure = $\mathbb{R}^n$, boundary is empty.
(e) Closed, interior empty, closure = boundary = $\mathbb{R}^{n-1}$.
(f) Neither open nor closed, interior empty, closure is $[0,1]$, boundary is $[0,1]$.
(g) Neither open nor closed, interior is $\{(x,y) \mid 0 < x < 1\}$, closure is $\{(x,y) \mid 0 \leq x \leq 1\}$, boundary is $\{(x,y) \mid x = 0 \text{ or } 1\}$.
(h) Closed, interior empty, closure = boundary = $\{x \in \mathbb{R}^n \mid \|x\| = 1\}$.
(i) If all cluster points appear in the sequence, then closed, otherwise neither. Interior empty, closure and boundary are the union of the sequence with all its cluster points.

□