MATH 3150 — HOMEWORK 2

Problem 1 (p. 97, #5)**.** Let x_n be a monotone increasing sequence bounded above and consider the set $S = \{x_1, x_2, \ldots\}$. Show that x_n converges to $\sup(S)$. Make a similar statement for decreasing sequences.

Remark. This shows that the *least upper bound property* — that every nonempty set with an upper bound has a least upper bound — implies the *monotone sequence property* — that every monotone increasing bounded sequence bounded above converges. Combined with the reverse implication proved in class, it follows that the least upper bound property is equivalent to completeness.

Solution. S is a set with an upper bound, so it has a supremum

$$x = \sup(S).$$

Let $\varepsilon > 0$. By our characterization of the supremum, there is some $x_N \in S$ such that $x - \varepsilon < x_N$ and since x_n is increasing it follows that

$$x - \varepsilon < x_n, \ \forall\, n \geq N$$
$$\implies |x_n - x| < \varepsilon \ \forall\, n \geq N$$

Thus $\lim_{n\to\infty} x_n = x = \sup(S)$.

If x_n is a decreasing sequence bounded below, then x_n converges to $\inf(\{x_n\})$ by a similar proof. □

Problem 2 (p. 97, #7)**.** For nonempty sets $A, B \subset \mathbb{R}$, let $A+B = \left\{x + y \;\middle|\; x \in A \text{ and } y \in B\right\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

Solution. Let $a = \sup(A)$ and $b = \sup(B)$. Then since $a \geq x$ for all $x \in A$ and $b \geq y$ for all $y \in B$, it follows that $a + b$ is an upper bound for $A + B$, i.e.

$$a + b \geq x + y, \ \forall\, x + y \in A + B.$$

Let $\varepsilon > 0$. Then there is some $x \in A$ and $y \in B$ such that $a - \varepsilon/2 < x$ and $b - \varepsilon/2 < y$, which means that

$$a + b - \varepsilon < x + y \in A + B,$$

and it follows that $a + b = \sup(A + B)$. □

Problem 3 (p. 52, #4)**.**

(a) Let x_n be a Cauchy sequence. Suppose that for every $\varepsilon > 0$ there is some $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$. Prove that $x_n \longrightarrow 0$.

(b) Show that the hypothesis that x_n be Cauchy in (a) is necessary, by coming up with an example of a sequence x_n which does not converge, but which has the other property: that for every $\varepsilon > 0$ there exists some $n > 1/\varepsilon$ such that $|x_n| < \varepsilon$.

Solution. (a) Let $\varepsilon > 0$ be given. Since x_n is Cauchy, there exists an N such that $|x_n - x_m| < \varepsilon/2$ for all $m, n \geq N$. If we now let

$$\varepsilon_1 = \min(\varepsilon/2, 1/N)$$

then it follows from the other assumption that there is a $k > 1/\varepsilon_1 \geq N$ such that

$$|x_k| < \varepsilon_1 \leq \varepsilon/2.$$

Thus, for $x_n \geq N$, we have

$$|x_n - 0| = |x_n - x_k + x_k| \leq |x_n - x_k| + |x_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

so that $x_n \longrightarrow x$.

(b) Consider the sequence

$$x_n = \begin{cases} 1 & n \text{ odd} \\ 1/n & n \text{ even.} \end{cases}$$

This clearly does not converge, and yet for any $\varepsilon > 0$ we can choose an even $n > 1/\varepsilon$ for which $|x_n| < \varepsilon$.

□

Problem 4 (p. 99 #15)**.** Let x_n be a sequence in $\mathbb{R}$ such that $|x_n - x_{n+1}| \leq \frac{1}{2}|x_{n-1} - x_n|$. Show that x_n is a Cauchy sequence.

Solution. To show that x_n is Cauchy, we must compare x_n and x_m for all $n, m \geq N$ for various N, not just subsequent elements. To do this we first note that for arbitrary $k > 0$,

$$\begin{aligned}
|x_n - x_{n+k}| &= |x_n - x_{n+1} + x_{n+1} - \cdots - x_{n+k-1} + x_{n+k-1} - x_{n+k}| \\
&\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{n+k-1} - x_{n+k}| \\
&\leq |x_n - x_{n+1}| + \tfrac{1}{2}|x_n - x_{n+1}| + \tfrac{1}{4}|x_n - x_{n+1}| + \cdots + \tfrac{1}{2^{k-1}}|x_n - x_{n+1}| \\
&= (1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^{k-1}})\,|x_n - x_{n+1}| \\
&< 2\,|x_n - x_{n+1}| \\
&\leq \frac{2}{2^n}|x_0 - x_1| = \frac{1}{2^{n-1}}|x_0 - x_1|.
\end{aligned}$$

Let $M = |x_0 - x_1| \in \mathbb{R}$. Then for an arbitrary $\varepsilon > 0$, we may choose N sufficiently large that $\frac{1}{2^{N-1}} < \frac{\varepsilon}{M}$. (This uses the fact that $1/2^n \longrightarrow 0$.) Thus for any $n, m \geq N$, supposing that $m \geq n$, we can write $m = n + k$ for some $k \geq 0$ and then

$$|x_n - x_m| = |x_n - x_{n+k}| < \frac{M}{2^{n-1}} \leq \frac{M}{2^{N-1}} < \varepsilon,$$

so x_n is Cauchy. □

Problem 5. Prove that an Archimedean ordered field in which every Cauchy sequence converges is complete (i.e. has the monotone sequence property). Here are some suggested steps:

(a) Denote the field by $\mathbb{F}$, and suppose x_n is a monotone increasing sequence bounded above by some $M \in \mathbb{F}$.

(b) Proceeding by contradiction, suppose x_n is not Cauchy. Deduce the existence of a subsequence $y_k = x_{n_k}$ with the property that

$$y_k \geq y_{k-1} + \varepsilon, \ \forall \ k \tag{1}$$

for some fixed positive number $\varepsilon > 0$ which does not depend on k.

(c) Using the Archimedean property, argue that y_k cannot be bounded above by M, hence obtaining a contradiction.

(d) Conclude that x_n converges.

Proof. Suppose $\mathbb{F}$ is Archimedean and has the property that every Cauchy sequence in $\mathbb{F}$ converges. Let x_n be a monotone sequence in $\mathbb{F}$, with an upper bound M, and suppose that x_n is not Cauchy. Then there exists an $\varepsilon > 0$ such that, for all $N \in \mathbb{N}$, there is a pair $n, m \geq N$ for which

$$|x_n - x_m| \geq \varepsilon.$$

(This is just the negation of the statement that x_n is Cauchy.)

We construct a subsequece as suggested by the hint. Choose $n_1 = 1$ (really it doesn't matter where you start), and by induction suppose that we have $n_1 < n_2 < \cdots < n_k$ such that $x_{n_k} \geq x_{n_{k-1}} + \varepsilon$. Set $N = n_k$; then by assumption there is a pair $n_{k+1}, m_{k+1} \geq n_k$ (and without loss of generality we can suppose that $n_{k+1} > m_{k+1}$) such that

$$\begin{aligned} \left|x_{n_{k+1}} - x_{m_{k+1}}\right| &\geq \varepsilon, \\ \implies x_{n_{k+1}} &\geq x_{m_{k+1}} + \varepsilon \\ &\geq x_{n_k} + \varepsilon \end{aligned}$$

since the sequence is increasing. This completes the induction step and gives a subsequence $y_k = x_{n_k}$ satisfying (1), where $\varepsilon > 0$ is a fixed positive number, per our assumption that x_n is not Cauchy.

Let $d = M - y_1$ be the distance from the first element of the subsequence to the upper bound for x_n. By the Archimedean property of $\mathbb{F}$, there exists some $N \in \mathbb{N}$ such that

$$N > d/\varepsilon, \iff \varepsilon N > d.$$

By the property (1) on the subsequence y_k, it follows that

$$y_N \geq y_1 + N\varepsilon > y_1 + d = M,$$

Since $y_N = x_{n_N}$ is an element of the original sequence, this contradicts the assumption that x_n is bounded.

Since we reached this conclusion by assuming that our bounded increasing sequence x_n was not Cauchy, it follows that x_n must be Cauchy, hence convergent by the assumption on $\mathbb{F}$. Since x_n was an arbitrary increasing bounded sequence, it follows that $\mathbb{F}$ has the monotone sequence property. ☐