Math 3150 Final Exam Fall 2014

Name: ____

- The exam will last 2 hours.
- There are 8 problems worth 12 points each.
- No notes or other study materials allowed.
- Use the back sides of the test pages for scratch work, or if you need extra space.

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Total	

Problem 1. Determine whether the following statements are **true** or **false**. If true, provide a proof; if false, provide a counterexample.

(a) Every Lipschitz function is uniformly continuous.

(b) Every Lipschitz function is differentiable.

(c) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then for any $a < b \in \mathbb{R}$, the set $f^{-1}([a, b])$ is bounded.

(d) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then for any $a < b \in \mathbb{R}$, the set $f([a, b])$ is closed.

Problem 2. Determine whether the following statements are **true** or **false**. If true, provide a proof; if false, provide a counterexample.

(a) If $A \subset \mathbb{R}^n$ is connected, then $\mathrm{bd}(A)$ is connected.

(b) If $A \subset \mathbb{R}^n$ is compact, then $\mathrm{bd}(A)$ is compact.

(c) If $A \subset \mathbb{R}^n$ is both compact and connected, then $\mathbb{R}^n \setminus A$ is connected.

(d) If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ are path-connected, then $A \times B \subset \mathbb{R}^{n+m}$ is path-connected.

Problem 3. Let $g(x) = f(x^3) + x$ where $f : [0,1] \longrightarrow \mathbb{R}$ is a differentiable function such that $f(0) = f(1)$. Show that there exists a point $c \in [0,1]$ such that $g'(c) = 1$.

Problem 4.

(a) Let $p(t) = t^3 + at^2 + bt + c$ be a cubic polynomial with real coefficients $a, b, c \in \mathbb{R}$. Use the Intermediate Value Theorem to show that p has a real root, i.e., there exists $t_0 \in \mathbb{R}$ such that $p(t_0) = 0$.

(b) What can you say about existence of real roots for a polynomial of arbitrary degree $k \in \mathbb{N}$?

Problem 5. Prove the following case of L'Hôpital's rule: Suppose $f : [a, b] \longrightarrow \mathbb{R}$ and $g : [a, b] \longrightarrow \mathbb{R}$ are differentiable at $x_0 \in (a, b)$, with $f(x_0) = g(x_0) = 0$ and $g'(x_0) \neq 0$. Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$

Problem 6. Consider the linear map $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, where $L(x, y) = (ax + by, cx + dy)$. In other words, L is given by multiplication by the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(a) Show that L is Lipschitz, and estimate a Lipschitz constant in terms of a, b, c, and d.

(b) Show that the set $\{L(x, y) : 0 \leq x, y \leq 1\} \subset \mathbb{R}^2$ is compact.

Problem 7. This problem concerns the integral $\int_0^1 x\,dx$.

(a) For a fixed n, let P_n be the partition $\{0 = x_0 < \cdots < x_n = 1\}$ of $[0, 1]$ into n equal intervals, so that $x_i = \frac{i}{n}$. Compute the upper and lower sums $U(x, [0, 1], P_n)$ and $L(x, [0, 1], P_n)$.

(b) Show that $\lim_{n\to\infty} U(x, [0, 1], P_n) = \lim_{n\to\infty} L(x, [0, 1], P_n) = \frac{1}{2}$.

(c) Using the results of parts (a) and (b), deduce that $f(x) = x$ is integrable on $[0, 1]$ and $\int_0^1 x\,dx = \frac{1}{2}$.

Problem 8. Consider the sequence f_k in the space $\mathcal{C}([-1,1];\mathbb{R})$ of continuous functions, where

$$f_k(x) = \frac{e^{kx}}{1+e^{kx}}.$$

Does f_k converge in $\mathcal{C}([-1,1];\mathbb{R})$?