MATH 3150 FINAL EXAM PRACTICE PROBLEMS – FALL 2013

Problem 1.

(a) Give an example of a connected set $A \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus A$ is not connected.
(b) Give an example of a compact set $K \subset \mathbb{R}^n$ which is not connected.

Solution.

(a) $A = \{x \in \mathbb{R}^n : 1 \leq \|x\| \leq 2\}$.
(b) $K = \{x \in \mathbb{R}^n : \|x\| \leq 1 \text{ or } 2 \leq \|x\| \leq 3\}$.

□

Problem 2. Let $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and let $G = \{(x, f(x)) : x \in A\} \subset \mathbb{R}^2$ be its graph.

(a) Show that $G \subset \mathbb{R}^2$ is closed.
(b) If A is path-connected, show that G is path-connected. (Updated to path-connected 10pm on 12/4).
(c) If A is compact, show that G is compact.

Solution.

(a) It suffices to show that if an arbitrary sequence in G converges, then its limit lies in G. Suppose then that $\big(x_k, f(x_k)\big) \longrightarrow (x, y)$. In particular $x_k \longrightarrow x$, and since f is continuous, $f(x_k) \longrightarrow f(x)$, so $y = f(x)$ and $(x, y) = \big(x, f(x)\big)$ is therefore a point in G.
(b) Let $\big(x, f(x)\big)$ and $\big(y, f(y)\big)$ be two points in G. Since A is path-connected, there exists a continuous path $\gamma : [a, b] \longrightarrow A$ such that $\gamma(a) = x$ and $\gamma(b) = y$, and then $\widetilde{\gamma} : [a, b] \longrightarrow \mathbb{R}^2$, $\widetilde{\gamma}(t) = \big(\gamma(t), f(\gamma(t))\big)$ is a path in G from $\big(x, f(x)\big)$ to $\big(y, f(y)\big)$.
(c) We use sequential compactness. Suppose $\big(x_k, f(x_k)\big)$ is a sequence in G, so in particular x_k is a sequence in A. Since A is sequentially compact, there is a convergent subsequence $x_{k_n} \longrightarrow x \in A$. Since f is continuous, $f(x_{k_n}) \longrightarrow f(x)$ so therefore $\big(x_{k_n}, f(x_{k_n})\big)$ is a convergent subsequence with limit $\big(x, f(x)\big) \in G$.

□

Problem 3. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$.

(a) If A and B are path connected, show that $A \times B \subset \mathbb{R}^{n+m}$ is path connected.
(b) If A and B are compact, show that $A \times B \subset \mathbb{R}^{n+m}$ is compact.

Solution.

(a) If (x_1, y_1) and (x_2, y_2) are points in $A \times B$, then there exist continuous paths $\gamma_1 : [a, b] \longrightarrow A$ and $\gamma_2 : [a, b] \longrightarrow B$ such that $\gamma_i(a) = x_i$, $\gamma_i(b) = y_i$ for $i = 1, 2$. Then

$$\widetilde{\gamma} : [a, b] \longrightarrow A \times B, \widetilde{\gamma}(t) = \big(\gamma_1(t), \gamma_2(t)\big)$$

is a continuous path from (x_1, y_1) to (x_2, y_2).
(b) See HW6, #5.

□

Problem 4. Show that $f(x) = x^2$ is uniformly continuous on the open interval $(-1, 2)$.

Solution. $f(x)$ is continuous, and restricted to a *closed* and bounded (hence compact) interval, say $[-1, 2]$, it is uniformly continuous. But then it is uniformly continuous on any subset thereof, such as $(-1, 2)$.

Alternatively, you can give a direct ε–δ proof. □

Problem 5. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Show that f is continuous, and uniformly continuous on $[-1, 1]$.
(b) Show that f is not differentiable at $x = 0$.

Solution.

(a) f is continuous at any $x \neq 0$ since there it is the product of the continuous function x and the composition of the continuous functions $\sin(x)$ and $1/x$.

At $x = 0$ we must show that

$$\lim_{x \to 0} f(x) = f(0) = 0.$$

We will show that $\lim_{x\to 0} |f(x)| = 0$ which implies the above.

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} |x| \left|\sin\left(\tfrac{1}{x}\right)\right| \leq \lim_{x\to 0} |x| = 0$$

since sin is bounded by 1 in absolute value.

Then f is uniformly continuous on any compact set, such as $[-1, 1]$ since this is true for any continuous function.

(b) The limit of the difference quotient

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} \sin\left(\tfrac{1}{x}\right)$$

does not exist. Therefore f is not differentiable at $x = 0$.

□

Problem 6. Let $f(x) = \int_0^{x^2} e^{\sqrt{t}}\, dt$ for $x \in [0, +\infty)$.

(a) Compute $f(0)$.
(b) Show that f is differentiable on $(0, +\infty)$ and compute $f'(x)$.

Solution.

(a) $f(0) = \int_0^0 e^{\sqrt{t}}\, dt = 0$ since the interval of integration has width 0.

(b) The integrand, $t \longmapsto e^{\sqrt{t}}$, is a continuous function on $[0, +\infty)$ and therefore by the fundamental theorem of calculus,

$$F(y) = \int_0^y e^{\sqrt{t}}\, dt$$

is differentiable with derivative $F'(y) = e^{\sqrt{y}}$. Now $f = F \circ g$ where $g(x) = x^2$, so using the chain rule,

$$f'(x) = e^{\sqrt{g(x)}} g'(x) = 2xe^x.$$ □

Problem 7. Define $f : [0,1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 2 & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2}. \end{cases}$$

Show that f is integrable and compute $\int_0^1 f(x)\,dx$.

Solution. It suffices to show that for any $\varepsilon > 0$, there exists a partition P of $[0,1]$ such that

$$U(f,[0,1],P) - L(f,[0,1],P) < \varepsilon$$

since this implies that $\sup_P \{L(f,[0,1],P)\}$ and $\inf_P \{U(f,[0,1],P)\}$ — the lower and upper integrals, respectively — are equal.

For any partition $P = \{0 = x_0 < x_1 < \cdots < x_N = 1\}$, the upper integral is

$$U(f,[0,1],P) = \sum_{i=1}^{N} \sup_{[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) = 2\sum_{i=1}^{N}(x_i - x_{i-1}) = 2,$$

and the lower integral is

$$L(f,[0,1],P) = \sum_{i=1}^{N} \inf_{[x_{i-1},x_i]} f(x)(x_i - x_{i-1}) = 2 \sum_{1/2 \notin [x_{i-1},x_i]} (x_i - x_{i-1}) + 0 \cdot (x_j - x_{j-1})$$

where $[x_{j-1}, x_j]$ is the interval of P which contains the point $x = \frac{1}{2}$. Thus

$$L(f,[0,1],P) = 2(1 - (x_j - x_{j-1})).$$

Thus given any $\varepsilon > 0$, we may choose a partition such that the interval $[x_{j-1}, x_j]$ containing $x = 1/2$ has width $(x_j - x_{j-1}) < \varepsilon/2$. For such a partition P,

$$U(f,[0,1],P) - L(f,[0,1],P) < 2 - 2(1 - \varepsilon/2) = \varepsilon.$$

Thus f is integrable. Then

$$\int_0^1 f(x)\,dx = \overline{\int_0^1} f(x)\,dx = 2$$

since the upper sums $U(f,[0,1],P)$ are all equal to 2 so their infimum is 2. □

Problem 8. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq C\,|x - y|^2, \quad \forall x, y \in \mathbb{R}$$

for some $C \geq 0$. Show that f must be constant. [Hint: show that it is differentiable first.]

Solution. By assumption, the limit

$$\lim_{x \to y} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{x \to y} C\,|x - y| = 0$$

which implies that

$$\lim_{x \to y} \frac{f(x) - f(y)}{x - y} = 0$$

and therefore f is differentiable at all $y \in \mathbb{R}$ with derivative $f'(y) = 0$. Since the derivative vanishes identically, f must be constant. □

Problem 9. Suppose $f : [0, +\infty) \longrightarrow \mathbb{R}$ is continuous and differentiable on $(0, +\infty)$, and suppose that

$$f(x) + x\,f'(x) \geq 0, \quad \forall x > 0.$$

Show that $f(x) \geq 0$ for all $x \geq 0$. [Hint: consider the function $g(x) = xf(x)$.]

Solution. If we define $g(x) = x\,f(x)$, then by the product rule,

$$g'(x) = f(x) + x\,f'(x) \geq 0, \quad \forall x > 0.$$

Thus g is increasing on $(0, +\infty)$. Furthermore,

$$g(0) = 0 \cdot f(0) = 0$$

so it follows that $g(x) \geq 0$ for $x > 0$.

Now, $f(x) = g(x)/x$ implies that $f(x) \geq 0$ for $x > 0$ since $1/x$ is positive there, and finally

$$\lim_{x \to 0} f(x) \geq 0$$

since the limit of a non-negative function is nonnegative. Thus $f(x) \geq 0$ for all $x \in [0, \infty)$. □