MATH 3150 FINAL EXAM PRACTICE PROBLEMS – FALL 2013

Problem 1.
(a) Give an example of a connected set $A \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus A$ is not connected.
(b) Give an example of a compact set $K \subset \mathbb{R}^n$ which is not connected.

Problem 2. Let $f : A \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and let $G = \{(x, f(x)) : x \in A\} \subset \mathbb{R}^2$ be its graph.
(a) Show that $G \subset \mathbb{R}^2$ is closed.
(b) If A is path-connected, show that G is path-connected.
(c) If A is compact, show that G is compact.

Problem 3. Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$.
(a) If A and B are path connected, show that $A \times B \subset \mathbb{R}^{n+m}$ is path connected.
(b) If A and B are compact, show that $A \times B \subset \mathbb{R}^{n+m}$ is compact.

Problem 4. Show that $f(x) = x^2$ is uniformly continuous on the open interval $(-1, 2)$.

Problem 5. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

(a) Show that f is continuous, and uniformly continuous on $[-1, 1]$.
(b) Show that f is not differentiable at $x = 0$.

Problem 6. Let $f(x) = \int_0^{x^2} e^{\sqrt{t}}\, dt$ for $x \in [0, +\infty)$.
(a) Compute $f(0)$.
(b) Show that f is differentiable on $(0, +\infty)$ and compute $f'(x)$.

Problem 7. Define $f : [0, 1] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 2 & x \neq \frac{1}{2} \\ 0 & x = \frac{1}{2}. \end{cases}$$

Show that f is integrable and compute $\int_0^1 f(x)\, dx$.

Problem 8. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq C\, |x - y|^2, \quad \forall x, y \in \mathbb{R}$$

for some $C \geq 0$. Show that f must be constant. [Hint: show that it is differentiable first.]

Problem 9. Suppose $f : [0, +\infty) \longrightarrow \mathbb{R}$ is continuous and differentiable on $(0, +\infty)$, and suppose that

$$f(x) + x\, f'(x) \geq 0, \quad \forall x > 0.$$

Show that $f(x) \geq 0$ for all $x \geq 0$. [Hint: consider the function $g(x) = x f(x)$.]