<u>Theorem</u> Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of differentiable functions on (a,b). Suppose that

(1) $f_n \longrightarrow f$ pointwise

(2) $f_n' \longrightarrow g$ uniformly

Then f is differentiable on (a,b) and $f' = g$.

<u>Proof</u> Let $\varepsilon > 0$. We want to show that f is differentiable at every $x_0 \in (a,b)$ and $f'(x_0) = g(x_0)$. For every $x \in (a,b)$, $x \neq x_0$, we have:

$$\left| \frac{f(x)-f(x_0)}{x-x_0} - g(x_0) \right| \leq \left| \frac{f(x)-f(x_0)}{x-x_0} - \frac{f_n(x)-f_n(x_0)}{x-x_0} \right| \quad (*)$$
$$+ \left| \frac{f_n(x)-f_n(x_0)}{x-x_0} - f_n'(x_0) \right| \quad (**)$$
$$+ \left| f_n'(x_0) - g(x_0) \right| \quad (***)$$

WLOG, we may assume $x > x_0$. By the Mean Value Theorem for the function $f_m - f_n$ (for any m, n), $\exists x_1 \in (x_0, x)$ such that

$$(1) \quad \frac{f_m(x)-f_m(x_0)}{x-x_0} - \frac{f_n(x)-f_n(x_0)}{x-x_0} = f_m'(x_1) - f_n'(x_1)$$

Now, $f_n' \xrightarrow{u} g$, so f_n' converges Cauchy uniformly to g. Hence, $\exists N_1$ s.t., for $m, n > N_1$,

$$(2) \quad \left| f_m'(x_1) - f_n'(x_1) \right| < \varepsilon/3$$

for all $x \in (a,b)$.

Using (1) and (2) at $x_1 \in (x_0, x)$, we find:

(3) $$\left| \frac{f_m(x) - f_n(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| < \varepsilon/3$$

We also know that $f_m \to f$, and so, letting $m \to \infty$ in (3), we obtain

(4) $$\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_n(x) - f_n(x_0)}{x - x_0} \right| < \varepsilon/3$$

for $n > N_1$

Using again the fact that $f_n' \to g$, we find an $N_2 \in \mathbb{N}$ s.t.

(5) $$\left| f'(x_0) - g(x_0) \right| < \varepsilon/3 \qquad \text{for } n > N_2$$

Choose $n > \max\{N_1, N_2\}$. Using the assumption that f_n is differentiable, we find that $\exists\, \delta > 0$ s.t.

(6) $$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - f_n'(x_0) \right| < \varepsilon/3$$

for $0 < |x - x_0| < \delta$.

Using now ~~[illegible]~~, together with (4), (5), (6), we conclude that

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| < \varepsilon \qquad \text{for } 0 < |x - x_0| < \delta$$

i.e., f is differentiable at x_0, and $f'(x_0) = g(x_0)$.

QED