Group Theory

Week #7, Lecture #28

REVIEW SESSION

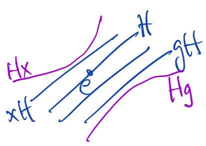
PLAN:
- I Groups & Subgroups
- II Cosets & Factor Groups
- III Homomorphisms & Isomorphisms
- IV Groups acting on sets
- V Sylow Theorems

I Groups & Subgroups

(1) Definitions, examples
(2) Normal subgroups
(3) Intersections, products, and direct products
(4) Commutator subgroup, normalizer, centralizer, center, ...
(5) Groups generated by sets (generators & relations)
(6) Orders of elements and subgroups — Lagrange's Theorem
(7) Subgroup lattice of a group

II Cosets and Factor Groups

(1) Equivalence relations associated to a subgroup $H \leq G$ yield left and right cosets of H in G: gH & Hg $(g \in G)$ which then partition G $\left(\begin{array}{l} x \sim y \iff x^{-1}y \in H \rightsquigarrow xH = yH \\ x \sim y \iff xy^{-1} \in H \quad Hx = Hy \end{array} \right)$



(left cosets are also right cosets) $\iff H \trianglelefteq G$

(2) Index of a subgroup: $[G:H] = \#\{\text{left cosets of } H \text{ in } G\}$
$= \#\{\text{right cosets of } H \text{ in } G\}$

Lagrange: (for G finite) $\boxed{[G:H] = \dfrac{|G|}{|H|}}$

Consequence: If $H \leq K \leq G$, then $[G:H] = [G:K] \cdot [K:H]$

$|G|/|H| = |G|/|K| \cdot |K|/|H|$

(3) If $N \trianglelefteq G$ is normal, then G/N is a group (factor group of G by N)
with operation: $xN * yN = xyN$
and $|G/N| = [G:N]$

eg: $G/G = \{e\}$
$G/\{e\} = G$

(4) If $G' = [G,G]$ is the commutator subgroup, then
$G_{ab} = G/G'$ is the abelianization of G.

eg: • If $H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\; a,b,c \in \mathbb{Z} \right\}$ (or $\mathbb{Z}_p$), then $H_{ab} \cong \mathbb{Z} \oplus \mathbb{Z}$ (or $\mathbb{Z}_p \oplus \mathbb{Z}_p$) & $H' = Z(H)$

• If G is simple (finite), then $G_{ab} = \begin{cases} G & , \text{ if } G \text{ abelian } (G = \mathbb{Z}_p) \\ 0 & , \text{ otherwise} \end{cases}$

(reason: $G' \trianglelefteq G$, so if G simple, then
$G' = \{e\} \Rightarrow G/G' = G$ or $G' = G \Rightarrow G/G' = \{e\}$)

[$G/G = \{$ left cosets of $G\} = \{gG : g \in G\} = \{G\}$]
trivial group

III Homomorphisms

① $\varphi : G \longrightarrow H$ hom if $\varphi(ab) = \varphi(a)\varphi(b)$
iso if φ hom & bijection
auto if $\varphi : G \to G$ iso
inner auto if $\exists g \in G$ s.t. $\varphi(x) = gxg^{-1}$, $\forall x \in G$
(i.e. φ is conjugation by g)

② Automorphism groups: $\mathrm{Aut}(G) = \{\varphi : G \to G \text{ auto}\} \leq \mathrm{Sym}(G)$
$\mathrm{Inn}(G) \cong G/Z(G)$

③ $\varphi : G \longrightarrow H$ hom $\rightsquigarrow \begin{cases} \ker(\varphi) = \{g \in G : \varphi(g) = e_H\} \trianglelefteq G \\ \mathrm{im}(\varphi) = \{h \in H : \exists g \in G, \varphi(g) = h\} \leq H \end{cases}$

Then: φ injective $\Longleftrightarrow \ker(\varphi) = \{e_G\}$
φ surjective $\Longleftrightarrow \mathrm{im}(\varphi) = H$

④ $\boxed{G/\ker(\varphi) \xrightarrow[\bar{\varphi}]{\cong} \mathrm{im}(\varphi)}$ $\quad \bar{\varphi}(g \cdot \ker(\varphi)) = \varphi(g)$ is a well-defined iso

⑤ $$\boxed{\begin{array}{l} H\cdot N/_H \cong H/_{H\cap N} \\ {}^{G/N}/_{H/N} \cong G/_H \end{array}}$$

(if $N \trianglelefteq G$ and $H \le G$)

(if $N \trianglelefteq G$, $H \trianglelefteq G$ and $N \le H$)

⑥ If $G \cong H$ (i.e., there is an iso $\varphi: G \longrightarrow H$), then all usual group properties are preserved

eg: . The lattices of subgroups are in bijection

. Being abelian, finite, simple, etc

. If groups are finite, the orders of the elements match (if $\varphi: G \longrightarrow H$ is iso, then $o(\varphi(x)) = o(x)$, $\forall x \in G$)

IV Groups acting on sets

① $G \circlearrowright S$ group G acting on set S by $(g,x) \mapsto g * x$ (or $g\cdot x$ or gx)

axioms: . $g*(h*x) = (gh)*x$ $\forall g,h \in G$, $\forall x \in S$

. $e*x = x$

$\leftrightsquigarrow$ $\varphi: G \longrightarrow Sym(S)$ $\varphi(g)(x) = gx$

homomorphism

action is faithful if φ injective, i.e., $\ker(\varphi) = \{e\}$ ($G_x = \{e\}$ $x \in S$)

② Orbits: $Gx := \{gx : g \in G\} \subset S$ $(\forall x \in S)$

Stabilizers: $G_x := \{g \in G : gx = x\} \le G$

Fixed point set: $S^G := \{x \in S : gx = x, \forall g \in G\} \subset S$

Facts: - orbits of G-action partition S

- stabilizers along an orbit are conjugate subgroups

- Orbit-Stabilizer Theorem: $\boxed{Gx \longleftrightarrow \{\text{left cosets of } G_x\}}$

$gx \longleftrightarrow gG_x$

③ Class Equation (for G, S finite) $$\boxed{|S| = |S^G| + \sum_{|Gx|>1} [G : G_x]}$$

④ Basic examples

(i) Trivial action: $gx = x \quad , \forall x \in S \longrightarrow \begin{cases} Gx = \{x\}, \; G_x = G \\ S^G = S \end{cases}$

(ii) Transitive actions: $\forall x, y \in S, \; \exists g \in G$ st $gx = y$

Then $Gx = S$

(iii) Left action of G on $S = G$: $(g, x) \longmapsto gx$

$Gx = G \qquad$ (action is transitive : $(\underbrace{yx^{-1}}_{g}) \cdot x = y$)

$G_x = \{e\} \qquad (gx = x \Rightarrow g = e)$

$G^G = \phi$

$\varphi : G \longrightarrow Sym(G)$ is injective

$(g \in \ker(\varphi) \Longleftrightarrow \varphi(g) = id \Longleftrightarrow g \cdot x = x \;\; \forall x \in G \Longleftrightarrow g = e)$

in particular; if $|G| = n$, then $\varphi : G \longrightarrow Sym(G) = S_n$ is inj

$\therefore$ [Cayley] $\boxed{G \text{ is isomorphic to a subgroup of } S_n}$

(iv) Conjugation action of G on $S = G \qquad g * x = gxg^{-1}$

Orbits: $Gx = \{gxg^{-1} : g \in G\} = Cl(x)$ conjugacy class of x

Stabilizers: $G_x = \{g : gxg^{-1} = x\} = C(x)$ centralizer of x

FPS: $S^G = \{x : gxg^{-1} = x\} = Z(G)$ center of G

Class eq: $$\boxed{|G| = |Z(G)| + \sum_{|Cl(x)| > 1} [G : C(x)]}$$

(v) Applications

- G is p-group $(G \neq \{e\}) \Rightarrow Z(G) \neq \{e\}$
- $|G| = p^2 \Rightarrow G$ is abelian , etc

(vi) Left action of G on left cosets of $H \leq G$:

$g \cdot xH = (gx)H$

- transitive $\Rightarrow$ single orbit
- stabilizer of $xH = N(H)$
- fixed point set $= \emptyset$

$$\left[\begin{array}{l} g\cdot xH = xH \\ \Leftrightarrow x^{-1}gx \in H \\ \Leftrightarrow g\in xHx^{-1} \\ \Leftrightarrow g\in N(H) \end{array}\right]$$

V Sylow Theorems (for a finite group G)

(1) Cauchy's Theorem: $\boxed{p \mid |G| \Rightarrow \exists x\in G \text{ s.t. } o(x) = p}$

(2) Sylow I: $\boxed{Syl_p(G) \neq \emptyset, \quad \forall p \mid |G|}$

here: • $P \leq G$ is a p-Sylow subgroup of G if P is a p-group (i.e. $|P|$ is a power of p) and $p \nmid [G:P]$

• $Syl_p(G) = \{P \leq G : P \text{ is a } p\text{-Sylow subgroup}\}$

(3) Sylow II: $\boxed{\text{All } p\text{-Sylow subgroups of } G \text{ are conjugate}}$
(i.e., the conjugation action of G on $Syl_p(G)$ is transitive)

(4) Sylow III: writing $n_p := |Syl_p(G)|$, and $|G| = p^k \cdot m$ with $p\nmid m$

we have: $\boxed{\begin{array}{l} \bullet\ n_p \equiv 1 \pmod p \\ \bullet\ n_p \mid m \end{array}}$

Note: • $n_p = 1 \Leftrightarrow Syl_p(G) = \{P\}$, and $P \trianglelefteq G$

• If all Sylow subgroups are normal, then $\boxed{G \cong \prod_{P\in Syl(P)} P}$

use the Decomposition Theorem, and:

• $\gcd(|H|, |K|) = 1 \Rightarrow H\cap K = \{e\}$

• If $N_1, N_2 \trianglelefteq G$ and $N_1 \cap N_2 = \{e\} \Rightarrow [N_1, N_2] = \{e\}$

$[\ g_i \in N_i \Rightarrow (g_1 g_2 g_1^{-1}) g_2^{-1} \in N_2$ and $g_1 (g_2 g_1^{-1} g_2^{-1}) \in N_1$, so $g_1 g_2 g_1^{-1} g_2^{-1} \in N_1 \cap N_2\ ]$

(5) Classification of finite abelian groups