Group Theory
Week #5, Lecture #17

GROUP ACTIONS ON SETS

<u>Definition</u> A (left) group action of a group G on a set S is a function $\mu : G \times S \longrightarrow S$, usually written as $\mu(g, x) = g * x = g \cdot x = gx$, such that:

(1) $\boxed{e * x = x}$ $\quad (\forall x \in S$, where e = identity of $G)$

(2) $\boxed{g * (h * x) = (g \cdot h) * x}$ $\quad (\forall g, h \in G, \forall x \in S)$

multiplication in G

schematic picture:

A group G can act on

(I) itself (in several ways)

(II) its subgroups (— " —) or their cosets

(III) on a vector space (by linear transformations)

(IV) on various geometric objects (graphs, polyhedra, manifolds, etc)

Examples

I <u>Ex 1</u> (G acting on itself by left multiplication)

$S = G \quad , \quad \mu : G \times G \longrightarrow G \quad \mu(g, x) = g \cdot x$

check: (1) $e \cdot x = x$ $\quad \leftarrow$ since e is the identity of G

(2) $g \cdot (h \cdot x) = ghx = (gh) \cdot x$ $\quad \leftarrow$ by associativity of group operation

<u>Ex 2</u> (G acting on itself by conjugation) !!

$S = G \quad , \quad \mu : G \times G \longrightarrow G \quad \mu(g, x) = g x g^{-1}$

check: (1) $e * x = e x e^{-1} = x$ ✓

(2) $g * (h * x) = g * (h x h^{-1}) = g (h x h^{-1}) g^{-1}$

$= (gh) x (h^{-1} g^{-1}) = (gh) x (gh)^{-1} = (gh) * x$ ✓

II **Ex 3** (G acting on its subgroups by conjugation)

$S = \{ H : H \leq G \}$ — the set of all subgroups of G

$\mu : G \times S \longrightarrow S$, $\mu(g, H) = gHg^{-1}$

Check: (1) $e * H = eHe^{-1} = H$

(2) $g*(h*H) = g*(hHh^{-1}) = g(hHh^{-1})g^{-1}$
$= (gh)H(gh)^{-1} = (gh)*H$ ✓

Ex 4 (G acting on the (left) cosets of a subgroup $H \leq G$)

$S = \{ xH : x \in G \}$ where $xH = yH \Leftrightarrow y^{-1}x \in H$

$\mu : G \times S \longrightarrow S$ $\quad g*(xH) = (gx)H$

Check: (1) $e*(xH) = (ex)H = xH$

(2) $g*(h*(xH)) = g*(hx \cdot H) = g(hx) \cdot H$
$= (gh) \cdot xH = (gh)*(xH)$

III **Ex 5** (G a linear group acting on a vector space by linear transformations)

Take $G = GL_n(F)$ (or one of its subgroups) where F is a field and $n \geq 1$

G acts on the F-vector vector space $S = F^n$ by

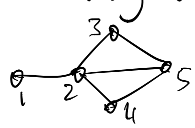
$\mu : G \times F^n \longrightarrow F^n$, $\mu(A, v) = A \cdot v$

A: $n \times n$ matrix $\quad$ $A \cdot v$: matrix mult.

check (1) $e = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $\quad I_n \cdot v = v$ ✓

(2) $A*(B*v) = A*(B \cdot v) = A \cdot (B \cdot v) = (A \cdot B) v = (A*B) v$ ✓

(IV) • Groups acting on graphs via graph automorphisms:

Γ:  $\downarrow \mu$ $\quad \Gamma = (V, E)$

V: vertices $\quad E$: edges

A graph automorphism is a bijection $\sigma: V \longrightarrow V$ which preserves the set of edges:

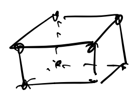
$\left[\begin{array}{l}\text{if } e=\{v,w\} \text{ is a edge in } \Gamma \quad (v,w \in V) \\ \text{then } \sigma(e):=\{\sigma(v),\sigma(w)\} \text{ is also an edge in } \Gamma.\end{array}\right.$

$$\boxed{\mathrm{Aut}(\Gamma) := \{\sigma \in \mathrm{Sym}(V) : \sigma \text{ is an auto of } \Gamma\}}$$

the automorphism group of Γ

(if $V=\{1,\dots,n\}$, then $\mathrm{Aut}(\Gamma) \leq S_n$)

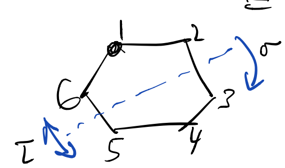
Exercise: Find $\mathrm{Aut}(\Gamma)$, where Γ is the "cubical graph"



$\Gamma \leq S_8$

- Groups acting on polyhedra, such as the regular n-gons, regular 3-dim polyhedra (tetrahedron, cube, icosahedron, ...), etc by symmetries

Example D_n = dihedral group of order $2n$
= group of symmetries of regular n-gon.



$g * v = g \cdot v$

g in D_n; v a vertex; $g \cdot v$ image of v under symmetry g

eg: $\sigma * 2 = \sigma(2) = 3$ $\qquad \sigma * 3 = \sigma(3) = 4$

$\tau * 2 = \tau(2) = 3$ $\qquad \tau * 3 = \tau(3) = 2$

- Groups acting on manifolds (or algebraic varieties, etc)

S = surface of genus 2

- Galois groups acting on field extensions (Studied in follow-up course on Rings & Fields – Galois Theory)

An alternative view of group actions is given by the Permutation Representations associated to them

$$\boxed{\left(\mu : G \times S \longrightarrow S \right) \rightsquigarrow \left(\varphi : G \longrightarrow \mathrm{Sym}(S) \right)}$$

group action of G on S — homomorphism from G to symmetric group on S

$$\boxed{\varphi(g)(x) = \mu(g,x)} \quad (*)$$

(i) The fact that μ satisfies axioms (1) & (2) implies that the function φ takes values in $\mathrm{Sym}(S)$ & φ is a hom

indeed: (a) Let $\sigma_g : S \to S$ be the function $x \longmapsto \sigma_g(x) = \mu(g,x) = g * x$ $(= \varphi(g)(x))$

Then σ_g is a bijection, $\boxed{\varphi(g) = \sigma_g}$ with inverse $\sigma_{g^{-1}}$:

$$\sigma_g \circ \sigma_{g^{-1}}(x) = \sigma_g(g^{-1} \cdot x) = g \cdot (g^{-1} x) \overset{(2)}{=} (g g^{-1}) x = e \cdot x \overset{(1)}{=} x$$

✓

(b) φ is a hom: $\varphi(gh) = \sigma_{gh} = \sigma_g \circ \sigma_h = \varphi(g) \circ \varphi(h)$

since: $\sigma_{gh}(x) = (gh) * x \overset{(2)}{=} g * (h * x) = \sigma_g \circ \sigma_h (x)$

(ii) Given a rep $\varphi : G \longrightarrow \mathrm{Sym}(S)$, define μ by the formula $(*)$, and check as above that μ satisfies (1) & (2), i.e., μ is an action of G on S.

Def. The action of G on S is called *faithful* if the associated perm. rep., $\varphi : G \to \mathrm{Sym}(S)$, is injective.

Equivalently: $\ker(\varphi) = \{e\}$, or:

$$\boxed{(gx = x \text{ for } \forall x \in S) \iff g = e}$$

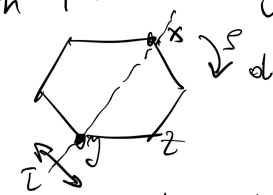
So, if G acts faithfully on S, then $G \xhookrightarrow{\varphi} \operatorname{im}(\varphi)$ is an isomorphism of G onto a subgroup of $\operatorname{Sym}(S)$.

In other words, G can be viewed as a subgroup of $\operatorname{Sym}(S)$.

Examples

(1) The action of D_n on $\{1, \dots, n\}$ coming from symmetries of the regular n-gon is faithful:

(If $g \in D_n$ fixes every vertex x of the n-gon then $g = e$)



fixes x and y, but not z

in fact, if g fixes 3 vertices, then $g = e$

$\therefore$ $\boxed{D_n \leq S_n}$

(2) $G = \mathbb{Z}_n$ acting on itself by (left) multiplication. Then the associated perm. rep,

$$\varphi : \mathbb{Z}_n \longrightarrow S_n$$

is again faithful ($g \cdot x = x \;\; \forall x \Rightarrow g = e$ true for every left action of G on G)

$\therefore$ $\boxed{\mathbb{Z}_n \leq S_n}$