**Real Analysis, MATH 3150** **Final Exam** **Fall 2023**

Please **print** your name ________________________________

You may use the blank, unnumbered, pages on the back of each numbered page for your work if needed. If you do this, be sure to note on the numbered page where the reader should look for the continuation of your work on the problem.
Cellphones and laptops must be turned off and placed on the floor.
For credit you need to fully justify your response to each question. You can cite results in the notes page that accompanies this test or by indicating a result in the text—for example, *since every bounded sequence contains a convergent subsequence, it follows that* ......

1. (11 pts) Prove that $x2^x = 9 - x^2$ for some $x \in (0, 2)$

2. (11 pts) Prove $|e^{-x} - e^{-y}| \leq |x - y|$ for all $x \geq 0, y \geq 0$.

3. (11 pts) Show that
$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin(n^2 x^3)$$
converges uniformly on $\mathbb{R}$ to a continuous function.

4. (10 pts) Find

$$\lim_{n\to\infty} \frac{1}{n}\big(\cos(1/n) + \cos(2/n) + \cdots + \cos((n-1)/n) + \cos(n/n)\big)$$

5. Let $f(x) = x\sin(x)$ for $x \in [-2, 2]$.

   (a) (5 pts) Write the Taylor polynomial of degree 4 for $f$ with center $a = 0$.

   (b) (5 pts) Give an upper bound for the error made in approximating the function $f(x)$ by the polynomial in part (a) for $x$ in the interval $[-2, 2]$.

6. Let $f$ be the function defined by

$$f(t) = \begin{cases} -2t & \text{for } t \leq 0 \\ \sin(t) & \text{for } 0 < t \leq \pi/2 \\ t - \pi/2 & \text{for } t > \pi/2 \end{cases}$$

(a) (5 pts) Determine $F(x) = \displaystyle\int_0^x f(t)\,dt$.

(b) (5 pts) Sketch the graph of $F$

(c) (2 pts) At which points, if any, is $F$ not continuous?

(d) (2 pts) At which points, if any, is $F$ not differentiable?

7. Let $f$ be the function defined on $[0, 1]$ by

$$f(t) = \begin{cases} 1 & \text{if } t = 1 - 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

(a) (5 pts) Prove that $f$ is integrable on $[0, 1]$

(b) (5 pts) Find the value of $\int_0^1 f(t)\, dt$

8. Let $f_n(x) = (x + \frac{1}{n})^2$ for $x \in [0, 2]$.

   (a) (5 pts) Does the sequence $(f_n)$ converge pointwise on $[0, 2]$? If so, find the limit function $f$.

   (b) (5 pts) Does $(f_n)$ converge uniformly on $[0, 2]$? Prove your assertion.

9. (a) (4 pts) Fix $a > 0$ and consider the power series $f_a(x) = \sum_{n\geq 1} \frac{1}{n}\left(\frac{x}{a}\right)^n$. Determine its radius of convergence $R$.

(b) (4 pts) Compute $f_a'(x)$ on $(-R, R)$, and identify this with a known function in closed form.

(c) (3 pts) Find an explicit expression for $f_a(x)$.

(d) (2 pts) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$.