1. (11 pts) Prove that $x2^x = 9 - x^2$ for some $x \in (0, 2)$

   Solution: Let $h(x) = x2^x - 9 + x^2$. Since h is a continuous function, $h(0) = 0(2^0) - 9 + 0^2 = -9 < 0$, and $h(2) = 2(2^2) - 9 + 2^2 = 8 - 9 + 4 = 3 > 0$, it follows from the Intermediate Value Theorem that $h(x) = 0$ for some $x \in (0, 2)$. Then for this value of x we have $x2^x - 9 + x^2 = 0$, and hence $x2^x = 9 - x^2$.

2. (11 pts) Prove $|e^{-x} - e^{-y}| \leq |x - y|$ for all $x \geq 0, y \geq 0$.

   Solution: Let $x \neq y$ be nonnegative numbers. $de^{-x}/dx = -e^{-x}$, so it follows from the Mean Value theorem that there is at least one number c between x and y such that

$$-e^{-c} = \frac{e^{-x} - e^{-y}}{x - y}$$

   c is nonnegative, so $-c \leq 0$ and hence $e^{-c} \leq 1$ since e^t is an increasing function and $e^0 = 1$. Thus,

$$1 \geq |e^{-c}| = \frac{|e^{-x} - e^{-y}|}{|x - y|}$$

   and hence $|x - y| \geq |e^{-x} - e^{-y}|$.

3. (11 pts) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin(n^2 x^3)$$

   converges uniformly on $\mathbb{R}$ to a continuous function.

   Solution: Since $|\sin(\theta)| \leq 1$ for all θ, we have that

$$\left| \frac{1}{n^{3/2}} \sin(n^2 x^3) \right| \leq \frac{1}{n^{3/2}} \quad \text{for all } n \in \mathbb{N} \text{ and } x \in \mathbb{R}$$

   $\sum 1/n^{3/2}$ is a convergent p-series with $p = 3/2 > 1$, so by the Weierstrass M-test it follows that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin(n^2 x^3)$ converges uniformly on $\mathbb{R}$ to a function f. From the theorem in the text that the uniform limit of continuous functions is continuous, it follows that f is continuous, and hence, the argument is complete.

4. (10 pts) Find

$$\lim_{n \to \infty} \frac{1}{n} \big(\cos(1/n) + \cos(2/n) + \cdots + \cos((n-1)/n) + \cos(n/n) \big)$$

   Solution: Let P_n be the partition of $[0, 1]$ given by

$$P_n = \{0 = t_0 < 1/n < 2/n < \cdots < (n-1)/n < n/n = 1 = t_n\}$$

Then

$$\begin{aligned}(1)\qquad &\frac{1}{n}\big(\cos(1/n)+\cos(2/n)+\cdots+\cos((n-1)/n)+\cos(n/n)\big)\\ &=\sum_{k=1}^{n}\cos(t_k)\cdot(t_k-t_{k-1})\end{aligned}$$

Note that the sum in equation (1) is a Riemann sum for $\cos(x)$, and the mesh of the partition P_n is $1/n$. Since $\cos(x)$ is continuous it follows that $\cos(x)$ is integrable on $[0,1]$. Moreover, since $\lim_{n\to\infty}\text{mesh}(P_n)=0$, it follows from the definition of the Riemann integral that the limit of the sums in equation (1) is $\int_0^1\cos(x)\,dx$. Since $d(\sin(x))/dx=\cos(x)$, it follows from the First Fundamental Theorem of Calculus that $\int_0^1\cos(x)\,dx=\sin(1)-\sin(0)=\sin(1)$. Hence the limit of the sums in equation (1) is $\sin(1)$.

5. Let $f(x)=x\sin(x)$ for $x\in[-2,2]$.

(a) (5 pts) Write the Taylor polynomial of degree 4 for f with center $a=0$.

Solution: The Taylor series around 0 for the sine function is

$$\sin(x)=x-x^3/3!+x^5/5!+\cdots+(-1)^{n+1}x^{2n+1}/(2n+1)!+\cdots.$$

Thus, the Taylor series around 0 for the function f is

$$x\sin(x)=x^2-x^4/3!+x^6/5!+\cdots+(-1)^{n+1}x^{2n+2}/(2n+1)!+\cdots.$$

Hence, the Taylor polynomial of degree 4 around 0 for the function f is

$$P_4(f,0)(x)=x^2-\frac{x^4}{6}.$$

Alternatively, we compute the successive derivatives of f and evaluate them at $x=0$:

$$\begin{aligned}f(x)&=x\sin(x) & f(0)&=0\\ f'(x)&=\sin(x)+x\cos(x) & f'(0)&=0\\ f''(x)&=2\cos(x)-x\sin(x) & f''(0)&=2\\ f^{(3)}(x)&=-x\cos(x)-3\sin(x) & f^{(3)}(0)&=0\\ f^{(4)}(x)&=x\sin(x)-4\cos(x) & f^{(4)}(0)&=-4\end{aligned}$$

and thus

$$\begin{aligned}P_4(f,0)(x)&=f(0)+f'(0)x+f''(0)x/2!+f^{(3)}(0)x/3!+f^{(4)}(0)x/4!\\ &=0+0\cdot x+2\frac{x^2}{2}+0\cdot x/6-4\frac{x^4}{24}\\ &=x^2-\frac{x^4}{6}.\end{aligned}$$

(b) (5 pts) Give an upper bound for the error made in approximating the function $f(x)$ by the polynomial in part (a) for x in the interval $[-2, 2]$.

Solution: From the first solution in part (a), we see that the Taylor series is a (converging) alternating series. Therefore, the error made in approximating the function f by the Taylor polynomial $P_4(x)$ is at most the absolute value of the first term omitted:

$$|f(x) - P_4(x)| \leq \frac{|x|^6}{5!}.$$

On the interval $[-2, 2]$, we have that $|x|^6 \leq 2^6$, and so

$$|f(x) - P_4(x)| \leq \frac{2^6}{120} = \frac{8}{15}.$$

Alternatively, we may proceed as in the first solution in part (a), and use Taylor's Theorem to approximate the remainder in the Taylor series. First, we compute $f^{(5)}(x) = x\cos(x) + 5\sin(x)$, and so

$$R_5(x) = \frac{f^{(5)}(y)}{5!}x^5 = (y\cos(y) + 5\sin(y))\frac{x^5}{120}.$$

for some y between 0 and x. Therefore, the error made is at most

$$\begin{aligned} |R_5(x)| &= \left|(y\cos(y) + 5\sin(y))\frac{x^5}{120}\right| \\ &\leq (|y|\,|\cos(y)| + 5\,|\sin(y)|)\frac{|x|^5}{120} \\ &\leq (2 \cdot 1 + 5 \cdot 1)\frac{2^5}{120} \\ &\leq 7\frac{|x|^5}{120}. \end{aligned}$$

On the interval $[-2, 2]$, this error is at most $\frac{28}{15}$, which is still an upper bound, but not as sharp as the one obtained by the first method.

6. Let f be the function defined by

$$f(t) = \begin{cases} -2t & \text{for } t \leq 0 \\ \sin(t) & \text{for } 0 < t \leq \pi/2 \\ t - \pi/2 & \text{for } t > \pi/2 \end{cases}$$

(a) (5 pts) Determine $F(x) = \int_0^x f(t)\,dt$.

Solution: For $t \leq 0$, we have

$$\begin{aligned} \int_0^x f(t)\,dt &= -\int_x^0 -2t\,dt \\ &= \int_x^0 2t\,dt \\ &= t^2\Big|_x^0 \\ &= x^2 \end{aligned}$$

where the first line follows from the definition that for $a < b$, we have $\int_b^a f = -\int_a^b f$ and the third line follows from the second line using the First Fundamental Theorem of Calculus and the property the $d(t^2)/dt = 2t$.

For $0 < x \leq \pi/2$ we have

$$\begin{aligned} \int_0^x f(t)\,dt &= \int_0^x \sin(t)\,dt \\ &= -\cos(t)\Big|_0^x \\ &= -\cos(x) - (-\cos(0)) \\ &= 1 - \cos(x) \end{aligned}$$

where the second line follows from the first line using the First Fundamental Theorem of Calculus and the property that $d(-\cos(t))/dt = \sin(t)$. The fourth line follows from the third line since $\cos(0) = 1$.

For $x > \pi/2$, we have

$$\begin{aligned} F(x) &= \int_0^x f(t)\,dt \\ &= \int_0^{\pi/2} f(t)\,dt + \int_{\pi/2}^x f(t)\,dt \\ &= 1 - \cos(\pi/2) + \int_{\pi/2}^x (t - \pi/2)\,dt \\ &= 1 + \left[\frac{t^2}{2} - \frac{\pi t}{2}\Big|_{\pi/2}^x\right] \\ &= 1 + \frac{x^2}{2} - \frac{\pi x}{2} - \frac{\pi^2}{8} + \frac{\pi^2}{4} \\ &= \frac{x}{2} \cdot (x - \pi) + 1 + \frac{\pi^2}{8} \end{aligned}$$

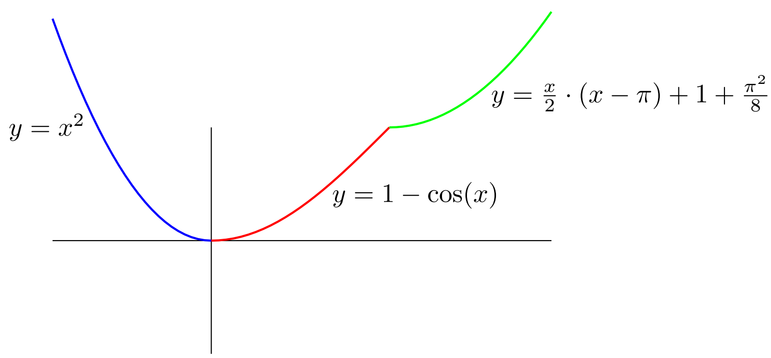
where the second line follows from the first line since $\int_a^b f + \int_b^c f = \int_a^c f$ for $a < b < c$. The fourth line follows from the third line using the First Fundamental Theorem of Calculus.

From the computations above, it follows that

$$F(x) = \begin{cases} x^2 & x \le 0 \\ 1 - \cos(x) & 0 < x \le \pi/2 \\ \frac{x}{2} \cdot (x - \pi) + 1 + \frac{\pi^2}{8} & \pi/2 < x \end{cases}$$

(b) (5 pts) Sketch the graph of F

Solution:



(c) (2 pts) At which points, if any, is F not continuous?

Solution: Since f is integrable, it follows from the first part of the Second Fundamental Theorem of Calculus that $F(x) = \int_0^x f$ is continuous for all x.

(d) (2 pts) At which points, if any, is F not differentiable?

Solution: From the second part of the Second Fundamental Theorem of Calculus it follows that F is differentiable at all values of x at which f is continuous. Thus, F is differentiable at all values of x except possibly at the point $x = \pi/2$ where f is not continous.

The limit of the slopes of secant lines from $(x, F(x))$ to $(\pi/2, 1)$ as x approches $\pi/2$ from values of x less than $\pi/2$ is the derivative of $1 - \cos(x)$ at $x = \pi/2$. That is

$$\lim_{x \to \pi/2^-} \frac{F(x) - F(\pi/2)}{x - \pi/2} = \left.\frac{d(1 - \cos(x))}{dx}\right|_{x=\pi/2} = \sin(\pi/2) = 1$$

Similarly, the limit of slopes of secant lines from $(\pi/2, 1)$ to $(x, F(X))$ as x approches $\pi/2$ from values of x greater than $\pi/2$ is the derivative of $\frac{x}{2} \cdot (x - \pi) + 1 + \frac{\pi^2}{8}$ at $x = \pi/2$. That is

$$\lim_{x \to \pi/2^+} \frac{F(x) - F(\pi/2)}{x - \pi/2} = \left. \frac{d(\frac{x}{2} \cdot (x - \pi) + 1 + \frac{\pi^2}{8})}{dx} \right|_{x=\pi/2}$$

$$= x - \pi/2\Big|_{x=\pi/2} = 0$$

Since $\lim_{x \to \pi/2^-} \frac{F(x)-F(\pi/2)}{x-\pi/2} \neq \lim_{x \to \pi/2^+} \frac{F(x)-F(\pi/2)}{x-\pi/2}$, it follows that F is not differentiable at $x = \pi/2$. This completes the proof that F is not differentiable only at $x = \pi/2$.

7. Let f be the function defined on $[0, 1]$ by

$$f(t) = \begin{cases} 1 & \text{if } t = 1 - 1/n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

(a) (5 pts) Prove that f is integrable on $[0, 1]$

Solution: From the inequality

$$0 \leq U(f) - L(f) \leq U(f, P) - L(f, P) \quad \text{for } P \text{ any partition of } [a, b]$$

and the Squeeze Lemma, it follows that in order to show $L(f) = U)f)$ it suffices to show that there is a sequence of paritions (P_n) of $[0, 1]$ such that $\lim_{n\to\infty}\big[U(f, P_n) - L(f, P_n)\big] = 0$.

This can be seen as follows. Note that every interval in a partition of $[0, 1]$ contains elements t with $f(t) = 0$, and since $f(t) \geq 0$ for all t, it follows that $L(f, P) = 0$ for all partitions P. Hence, $L(f) = 0$, and it suffices to show there is a sequence of partitions (P_n) with $\lim_{n\to\infty} U(f, P_n) = 0$.

Next note that from the definition of f, it follows that $U(f, P)$ is the sum of the lengths of the intervals in the partition P that contain an element of the form $1/k$ for some $k \in \mathbb{N}$. Hence, it suffices to show that given $n \in \mathbb{N}$ there is a partition of $[0, 1]$ such that the limit as n goes to infinity of the sum of the lengths of the subintervals that contain an element of the form $1 - 1/k$ with $k \in \mathbb{N}$ is zero. This can be done as follows.

We will use following formula for the distance between $1 - 1/(k - 1)$ and $1 - 1/k$ is

$$\begin{aligned} 1 - 1/k - (1 - 1/(k - 1)) &= 1/(k - 1) - 1/k \\ &= [k - (k - 1)]/k(k - 1) \\ &= 1/k(k - 1) \end{aligned} \tag{2}$$

Note that for $n \in \mathbb{N}$, the are n points t in the interval with $t \leq 1 - 1/n$ and $f(t) = 1$. The points are $0, 1 - 1/2, 1 - 1/3, \ldots, 1 - (n-1), 1 - 1/n$.

Define the partition P_n of $[0,1]$ as follows. $t_0 = 0$, $t_1 = 1/3n(n-1)$, for $2 \leq k \leq n-1$ let the elements $\ell_k = 1 - 1/k - 1/3n(n-1)$ and $r_k = 1 - 1/k + 1/3n(n-1)$ be in the partition, along with $1 - 1/n$ and 1. From equation (2) it follows that $r_k < \ell_{k+1}$ so the endpoints of the intervals in the partition are

$$0, 1/3n(n-1), \ell_2, r_2, \ell_3, r_3, \ldots, \ell_{n-1}, r_{n-1}, 1 - 1/(n-1), 1$$

The intervals in the partition that contain an element t with $f(t) = 1$ are $[0, 1/3n(n-1)]$, $[\ell_k, r_k]$ for $2 \leq kr_{n-1}$, and $[1 - 1/n, 1]$. The lengths of these intervals are $1/3n(n-1), 2/3n(n-1)$, and $1/n$. There are $n-2$ intervals of length $2/3n(n-1)$ so the sum of the lengths of the subintervals that contain a t with $f(t) = 1$ is

$$\begin{aligned} U(f, P_n) &= \frac{1}{3n(n-1)} + (n-2) \cdot \left(\frac{2}{3n(n-1)} \right) + \frac{1}{n} \\ &= \frac{1}{3n(n-1)} + \frac{2(1-2/n)}{3n(1-1/n)} + \frac{1}{n} \end{aligned}$$

Thus,

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \left[\frac{1}{3n(n-1)} + \frac{2(1-2/n)}{3n(1-1/n)} + \frac{1}{n} \right] = 0 + 0 + 0$$

and the proof is complete.

(b) (5 pts) Find the value of $\displaystyle\int_0^1 f(t)\, dt$

Solution: From the above, we conclude that

$$\int_0^1 f(t)\, dt = L(f) = U(f) = 0.$$

8. Let $f_n(x) = (x + \frac{1}{n})^2$ for $x \in [0,2]$.

(a) (5 pts) Does the sequence (f_n) converge pointwise on $[0,2]$? If so, find the limit function f.

Solution: For each $x \in [0,2]$, we have (by properties of limits of converging sequences),

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(x + \frac{1}{n} \right)^2 = \left(x + \lim_{n\to\infty} \frac{1}{n} \right)^2 = (x+0)^2 = x^2.$$

Therefore, the sequence (f_n) converges pointwises on the interval $[0,2]$ to the function $f(x) = x^2$.

(b) (5 pts) Does (f_n) converge uniformly on $[0,2]$? Prove your assertion.

Solution: We have

$$\begin{aligned}|f_n(x) - f(x)| &= \left|\left(x + \frac{1}{n}\right)^2 - x^2\right| \\ &= \left|2\frac{x}{n} + \frac{1}{n^2}\right| \\ &\leq \frac{4}{n} + \frac{1}{n^2}.\end{aligned}$$

Since $\lim_{n\to\infty}(\frac{4}{n} + \frac{1}{n^2}) = 0$, independently of x, we have that

$$0 \leq \inf\{|f_n(x) - f(x)| : x \in [0,2], n \in \mathbb{N}\} \leq \lim_{n\to\infty}\left(\frac{4}{n} + \frac{1}{n^2}\right) = 0,$$

and so (f_n) converge uniformly to f on $[0,2]$.

9. (a) (4 pts) Fix $a > 0$ and consider the power series $f_a(x) = \sum_{n\geq 1} \frac{1}{n}(\frac{x}{a})^n$. Determine its radius of convergence R.

Solution: The limit of the absolute values of consecutive terms in the series is equal to

$$\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}(\frac{x}{a})^{n+1}}{\frac{1}{n}(\frac{x}{a})^n}\right| = \lim_{n\to\infty}\frac{n}{n+1}\frac{|x|}{a} = \frac{|x|}{a}.$$

By the Ratio Test, we know this converges when $\frac{|x|}{a} < 1$, that is, $|x| < a$. Therefore, the radius of convergence is $R = a$.

(b) (4 pts) Compute $f_a'(x)$ on $(-R, R)$, and identify this with a known function in closed form.

Solution: Differentiating the power series $f_a(x)$ term by term (within its radius of convergence), and summing up the resulting geometric series (with initial term 1 and ratio x/a), we obtain

$$\begin{aligned}f_a'(x) &= \sum_{n\geq 1}\frac{1}{a^n}x^{n-1} \\ &= \frac{1}{a}\sum_{k\geq 0}\left(\frac{x}{a}\right)^k \\ &= \frac{1}{a}\frac{1}{1-\frac{x}{a}} \\ &= \frac{1}{a-x}.\end{aligned}$$

(c) (3 pts) Find an explicit expression for $f_a(x)$.

Solution: Using the Fundamental Theorem of Calculus (part I), we find that

$$f_a(x) = \int f_a'(x)dx = \int \frac{1}{a-x}dx = -\log(a-x) + C$$

for some constant C. But

$$f_a(0) = \sum_{n\geq 1} \frac{1}{n}\left(\frac{0}{a}\right)^n = 0$$

and so $C = f_a(0) + \log(a-0) = \log(a)$. Hence,

$$f_a(x) = -\log\left(1 - \frac{x}{a}\right).$$

(d) (2 pts) Evaluate the series $\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$.

Solution: The series is of the type in part (a), with $a = 3$ and $x = 1$. Using the formula for $f_a(x)$ obtained in part (c), we get:

$$\sum_{n=1}^{\infty} \tfrac{1}{n\,3^n} = f_3(1) = -\log\left(1 - \frac{1}{3}\right) = \log\left(\frac{3}{2}\right).$$