Hyperplane arrangements with trivial algebraic monodromy

Alexandru Suciu

Northeastern University

Hyperplane Arrangements 2023

Rikkyo University, Tokyo, Japan

December 13, 2023

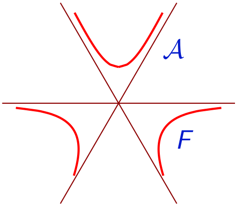
1. Milnor fibrations of arrangements
   - Milnor fibrations
   - The Milnor fiber as a cover
2. Milnor fibrations with trivial algebraic monodromy
   - Trivial algebraic monodromy
   - Abelian duality and propagation of jump loci
   - Lie algebras associated to groups
   - Lie algebras of arrangements and Milnor fibers
3. Examples
   - Decomposable arrangements
   - Falk's pair of arrangements
   - The deleted B_3 arrangement
   - Yoshinaga's icosidodecahedral arrangement

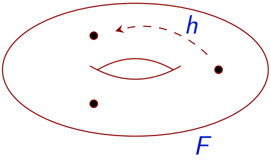
Milnor fibrations

- Let $\mathcal{A}$ be a central arrangement of n hyperplanes in $\mathbb{C}^{d+1}$, and fix an ordering on $\mathcal{A}$.
- To each hyperplane $H \in \mathcal{A}$, we may associate a multiplicity $m_H \in \mathbb{N}$. This yields a multi-arrangement $(\mathcal{A}, \mathrm{m})$, where $\mathrm{m} = (m_H)_{H \in \mathcal{A}} \in \mathbb{N}^n$.
- For each $H \in \mathcal{A}$, let f_H be a linear form with $\ker(f_H) = H$. Then

$$f_{\mathrm{m}} = \prod_{H \in \mathcal{A}} f_H^{m_H}$$

  is a homogeneous polynomial of degree $N = \sum_{H \in \mathcal{A}} m_H$.
- Let $M = \mathbb{C}^{d+1} \setminus \bigcup_{H \in \mathcal{A}} H$ be the complement of $\mathcal{A}$.

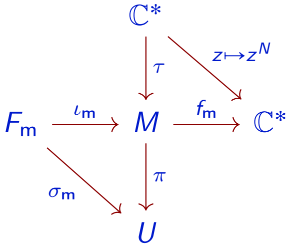




- The polynomial map $f_{\mathrm{m}}\colon \mathbb{C}^{d+1} \to \mathbb{C}$ restricts to a smooth fibration, $f\colon M \to \mathbb{C}^*$, called the *Milnor fibration* of $(\mathcal{A}, \mathrm{m})$.
- The *Milnor fiber* is $F_{\mathrm{m}} := f_{\mathrm{m}}^{-1}(1)$. The monodromy, $h\colon F_{\mathrm{m}} \to F_{\mathrm{m}}$, is given by $h(z) = e^{2\pi i/N} z$.
- F_{m} is a Stein manifold. It has the homotopy type of a finite CW-complex of dimension d. It is connected iff $\gcd(\mathrm{m}) = 1$.
- When all $m_H = 1$, the polynomial $f = f_{\mathrm{m}}$ is the usual defining polynomial of $\mathcal{A}$ and $F = F_{\mathrm{m}}$ is the usual Milnor fiber.

The Milnor fiber as a cover

- Let $U = \mathbb{P}(M)$. Get commuting diagram with row the Milnor fibration and column the Hopf fibration.



- $(f_{\mathrm{m}})_{\sharp} \colon \pi_1(M) \to \pi_1(\mathbb{C}^*)$ sends each meridional generator γ_H to m_H.
- It follows that $\sigma_{\mathrm{m}} \colon F_{\mathrm{m}} \to U$ is the regular, $\mathbb{Z}_N$-cover classified by the epimorphism $\pi_1(U) \twoheadrightarrow \mathbb{Z}_N$, $\bar{\gamma}_H \mapsto m_H$.

Trivial algebraic monodromy

- We say that $(\mathcal{A}, \mathrm{m})$ has *trivial algebraic monodromy* (over $\Bbbk = \mathbb{Z}$ or $\Bbbk = \mathbb{Q}$) if $h_*\colon H_*(F_{\mathrm{m}};\Bbbk) \to H_*(F_{\mathrm{m}};\Bbbk)$ is the identity.
- Let $\Delta_q(t) = \det(t \cdot \mathrm{id} - h_q)$ be the characteristic polynomial of $h_q\colon H_q(F_{\mathrm{m}};\mathbb{Q}) \to H_q(F_{\mathrm{m}};\mathbb{Q})$. Then

$$\Delta_q(t) = (t-1)^{b_q(U)} \cdot \prod_{1<k|N} \Phi_k(t)^{\mathrm{depth}_q(\rho_{\mathrm{m}}^{N/k})}.$$

  where $\rho_{\mathrm{m}}\colon \pi_1(U) \to \mathbb{C}^*$ is the character $\overline{\gamma}_H \mapsto e^{2\pi \mathrm{i}\, m_H/N}$ and $\mathrm{depth}_q(\rho) := \dim_{\mathbb{C}} H_q(U;\mathbb{C}_\rho)$.

- Hence

$$h_q \text{ is the identity} \iff b_q(F_{\mathrm{m}}) = b_q(U) \iff \Delta_q(t) = (t-1)^{b_q(F_{\mathrm{m}})}.$$

Characteristic varieties

- Let $\mathcal{V}^q_s(X) := \{\rho \in H^1(X;\mathbb{C}^*) \mid \dim_{\mathbb{C}} H_q(X;\mathbb{C}_\rho) \geqslant s\}$ and $\mathcal{W}^q_s(X) := \mathcal{V}^q_s(X) \cap H^1(X;\mathbb{C}^*)^0$.
- The varieties $\mathcal{V}^1_s(X)$ depend only on $\pi_1(X)/\pi_1(X)''$.
- If X is a smooth, complex quasi-projective variety, then each irreducible component of $\mathcal{V}^q_s(X)$ is a torsion-translated subtorus.
- The map $\pi^*\colon H^1(U;\mathbb{C}^*) \hookrightarrow H^1(M;\mathbb{C}^*)$ restricts to isomorphisms $\mathcal{V}^1_s(U) \xrightarrow{\simeq} \mathcal{V}^1_s(M)$, $\forall s \geqslant 1$, $\mathcal{V}^q_1(U) \cup \mathcal{V}^{q-1}_1(U) \xrightarrow{\simeq} \mathcal{V}^q_1(M)$, $\forall q \geqslant 1$.
- All positive-dimensional components of $\mathcal{V}^1_1(M)$ passing through $1 \in H^1(M;\mathbb{C}^*)$ arise from multinets on sub-arrangements of $\mathcal{A}$.
- If $\mathcal{V}^1_1(M)$ has no essential components, then the algebraic monodromy $h_*\colon H_1(F;\mathbb{Q}) \to H_1(F;\mathbb{Q})$ is trivial.
- If $\mathcal{A}$ admits a reduced multinet, then $h_*\colon H_1(F;\mathbb{Q}) \to H_1(F;\mathbb{Q})$ is non-trivial.

Proposition (Dimca–Papadima 2011)

The morphism $\sigma_{\mathrm{m}}^{*}\colon H^1(U;\mathbb{C}^*) \to H^1(F_{\mathrm{m}};\mathbb{C}^*)$ *restricts to maps* $\mathcal{V}^q_s(U) \to \mathcal{V}^q_s(F_{\mathrm{m}})$, *for all* $q, s \geqslant 1$.

The following theorem strengthens another one of their results.

Theorem

Suppose $H_1(F_{\mathrm{m}};\mathbb{Q}) \to H_1(F_{\mathrm{m}};\mathbb{Q})$ *is the identity. Then the morphism* $\sigma_{\mathrm{m}}^{*}\colon H^1(U;\mathbb{C}^*) \to H^1(F_{\mathrm{m}};\mathbb{C}^*)^0$ *is a surjection with* $\ker(\sigma_{\mathrm{m}}^{*}) \cong \mathbb{Z}_N$. *Moreover,*

(1) *For each* $s \geqslant 1$, *the map* σ_{m}^{*} *establishes a bijection between the irreducible components of* $\mathcal{V}^1_s(U)$ *and* $\mathcal{W}^1_s(F_{\mathrm{m}})$ *that pass through* 1.

(2) *The map* $\sigma_{\mathrm{m}}^{*}\colon \mathcal{V}^1_1(U) \to \mathcal{W}^1_1(F_{\mathrm{m}})$ *is a surjection.*

Abelian duality and propagation of jump loci

Let X be a connected, finite CW-complex, $G = \pi_1(X)$.

Definition (Denham–S.–Yuzvinsky 2016/17)

X is an ab-*duality space* of dimension n if $H^i(X, \mathbb{Z}G_{\text{ab}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}G_{\text{ab}}) \neq 0$ and torsion-free.

Theorem (DSY)

Let X be an abelian duality space of dimension n. Then:

- $b_1(X) \geqslant n - 1$.
- $b_i(X) \neq 0$, *for* $0 \leqslant i \leqslant n$ *and* $b_i(X) = 0$ *for* $i > n$.
- $(-1)^n \chi(X) \geqslant 0$.
- *The characteristic varieties "propagate":* $\mathcal{V}^1_1(X) \subseteq \cdots \subseteq \mathcal{V}^n_1(X)$.
- *Hence, if X is formal, then the resonance varieties also propagate:* $\mathcal{R}^1_1(X) \subseteq \cdots \subseteq \mathcal{R}^n_1(X)$.

- Set $G_{\mathrm{abf}} = G_{\mathrm{ab}} / \operatorname{Tors}$.
- We say X is an abf-*duality space* of dimension n if $H^i(X, \mathbb{Z}G_{\mathrm{abf}}) = 0$ for $i \neq n$ and $H^n(X, \mathbb{Z}G_{\mathrm{abf}}) \neq 0$ and torsion-free.
- In this case, $\mathcal{W}^1_1(X) \subseteq \cdots \subseteq \mathcal{W}^n_1(X)$.
- Let $F \to E \to B$ be a fibration. We say it is an ab-*exact fibration* if the sequence $0 \to \pi_1(F)_{\mathrm{ab}} \to \pi_1(E)_{\mathrm{ab}} \to \pi_1(B)_{\mathrm{ab}} \to 0$ is exact.
- (DSY) Let $F \to E \to B$ be an ab-exact fibration. If E and B are ab-duality spaces of dimensions m and n and if $\dim F = m - n$, then F is an ab-duality space of dimension $m - n$.
- Let $F \to E \to B$ be an abf-exact fibration. If E and B are abf-duality spaces of dimensions m and n and if $\dim F = m - n$, then F is an abf-duality space of dimension $m - n$.

Theorem (DSY)

Let $\mathcal{A}$ be a central arrangement of rank r. Then $M(\mathcal{A})$ is an abelian duality space of dimension r.

Corollary

1. *If the monodromy action on $H_1(F_{\mathrm{m}};\mathbb{Z})$ is trivial, then F_{m} is an* ab*-duality space of dim $r-1$ and $\mathcal{V}^1_1(X) \subseteq \cdots \subseteq \mathcal{V}^{r-1}_1(X)$.*
2. *If the monodromy action on $H_1(F_{\mathrm{m}};\mathbb{Q})$ is trivial, then F_{m} is an* abf*-duality space of dim $r-1$ and $\mathcal{W}^1_1(X) \subseteq \cdots \subseteq \mathcal{W}^{r-1}_1(X)$.*

Proposition

1. *If $h_*\colon H_i(F_{\mathrm{m}};\mathbb{Q}) \to H_i(F_{\mathrm{m}};\mathbb{Q})$ is the identity for $i \leqslant q$, then F_{m} is q-formal, and $\mathcal{R}^1_1(X) \subseteq \cdots \subseteq \mathcal{R}^q_1(X)$.*
2. *If $h_*\colon H_i(F_{\mathrm{m}};\mathbb{Q}) \to H_i(F_{\mathrm{m}};\mathbb{Q})$ is the identity for $i < r$, then F_{m} is formal, and $\mathcal{R}^1_1(X) \subseteq \cdots \subseteq \mathcal{R}^{r-1}_1(X)$.*

Lower central series

- The *lower central series* of a group G is defined inductively by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- Here, if $H, K < G$, then $[H, K]$ is the subgroup of G generated by $\{[a, b] := aba^{-1}b^{-1} \mid a \in H, b \in K\}$. If $H, K \triangleleft G$, then $[H, K] \triangleleft G$.
- The subgroups $\gamma_k(G)$ are, in fact, characteristic subgroups of G. Moreover, $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, $\forall k, \ell \geqslant 1$.
- In particular, it is a *central* series, i.e., $[G, \gamma_k(G)] \subseteq \gamma_{k+1}(G)$.
- It is also a *normal* series, i.e., $\gamma_k(G) \triangleleft G$. Each quotient,

  $$\operatorname{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$$

  lies in the center of $G/\gamma_{k+1}(G)$, and thus is an abelian group.
- If G is finitely generated, then so are its LCS quotients. Set $\phi_k(G) := \operatorname{rank} \operatorname{gr}_k(G)$.

Associated graded Lie algebras

- $\operatorname{gr}(G) := \bigoplus_{k\geqslant 1} \operatorname{gr}_k(G)$ is a graded Lie algebra (over $\mathbb{Z}$), with addition induced by the group multiplication and with Lie bracket $[\, ,\,]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by the group commutator.
- $G/\gamma_k(G)$ is the maximal $(k-1)$-step nilpotent quotient of G. $G/\gamma_2(G) = G_{\text{ab}}$, while $G/\gamma_3(G) \leftrightarrow H^{\leqslant 2}(G;\mathbb{Z})$.
- The *Chen Lie algebra* of G is defined as $\operatorname{gr}(G/G'')$. We have a surjection $\operatorname{gr}_k(G) \twoheadrightarrow \operatorname{gr}_k(G/G'')$, which is an iso for $k \leqslant 3$.
- Assuming G is finitely generated, write $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$ for the *Chen ranks*. We have $\phi_k(G) \geqslant \theta_k(G)$, with equality for $k \leqslant 3$.
- Example: if F_n is the free group of rank n, then
  - $\operatorname{gr}(F_n)$ is the free Lie algebra $\operatorname{Lie}(\mathbb{Z}^n)$.
  - $\operatorname{gr}_k(F_n)$ is free abelian, $\phi_k(F_n) = \frac{1}{k}\sum_{d\mid k} \mu(d) n^{k/d}$.
  - $\operatorname{gr}_k(F_n/F_n'')$ is free abelian, $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, for $k \geqslant 2$.

Holonomy Lie algebra

- For G finitely generated group, define $\mathfrak{h}(G) := \mathrm{Lie}(H)/\mathrm{ideal}(\mathrm{im}(\nabla_G))$, where $H = G_{\mathrm{abf}}$, $\mathrm{Lie}(H)_1 = H$, $\mathrm{Lie}(H)_2 = H \wedge H$, and ∇_G is the dual of $\cup_G \colon H^1(G;\mathbb{Z}) \wedge H^1(G;\mathbb{Z}) \to H^2(G;\mathbb{Z})$.
- $\mathfrak{h}(G)$ is a quadratic Lie algebra, determined solely by $H^{\leqslant 2}(G)$.
- There is a natural epi $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$ which restricts to isos in degrees 1 and 2, and factors through epi $\mathfrak{h}(G)/\mathfrak{h}(G)'' \twoheadrightarrow \mathrm{gr}(G/G'')$.
- G is 1-*formal* if the Malcev Lie algebra $\mathfrak{m}(G) := \mathrm{Prim}(\widehat{\mathbb{Q}[G]})$ is isomorphic to the LCS-completion completion of $\mathfrak{h}(G) \otimes \mathbb{Q}$.
- In that case, $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$ and $\mathfrak{h}(G)/\mathfrak{h}(G)'' \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G/G'') \otimes \mathbb{Q}$.

Lie algebras of arrangement groups

- (Kohno 1983) The holonomy Lie algebra of $G = G(\mathcal{A})$ is determined by $L_{\leqslant 2}(\mathcal{A})$:

$$\mathfrak{h}(G) = \mathrm{Lie}(x_H : H \in \mathcal{A}) \Big/ \mathrm{ideal}\left\{ \Big[x_H, \sum_{\substack{K \in \mathcal{A} \\ K \supset Y}} x_K \Big] \ : \ \substack{H \in \mathcal{A}, Y \in L_2(\mathcal{A}) \\ H \supset Y} \right\}.$$

- Since M is formal, G is 1-formal. Hence, $\mathrm{gr}(G) \otimes \mathbb{Q} \cong \mathfrak{h}(G) \otimes \mathbb{Q}$ is determined by $H^{\leqslant 2}(M; \mathbb{Q})$, and thus, by $L_{\leqslant 2}(\mathcal{A})$.
- (Papadima–S. 2004) $L_{\leqslant 2}(\mathcal{A})$ also determines the Chen ranks $\theta_k(G)$.
- Explicit combinatorial formulas for the LCS ranks $\phi_k(G)$ are known in some cases, but not in general.
- (Porter–S. 2020) The map $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism, but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- There can be torsion in $\mathrm{gr}(G)$ (S. 2001), but this torsion may not be combinatorially determined (A-B, G-B, V-S 2020).

Trivial algebraic monodromy and Lie algebras

Theorem

Suppose $h_*\colon H_1(F_{\mathrm{m}};\mathbb{Z}) \to H_1(F_{\mathrm{m}};\mathbb{Z})$ *is the identity. Then*

- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}})) \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M))$.
- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}})/\pi_1(F_{\mathrm{m}})'') \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M)/\pi_1(M)'')$.

Theorem

Suppose $h_*\colon H_1(F;\mathbb{Q}) \to H_1(F;\mathbb{Q})$ *is the identity. Then*

- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}}) \otimes \mathbb{Q} \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M)) \otimes \mathbb{Q}$.
- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}}/\pi_1(F''_{\mathrm{m}}) \otimes \mathbb{Q} \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M)/\pi_1(M)'') \otimes \mathbb{Q}$.

Hence, $\phi_k(\pi_1(F_{\mathrm{m}}) = \phi_k(\pi_1(M))$ *and* $\theta_k(\pi_1(F_{\mathrm{m}}) = \theta_k(\pi_1(M))$, $\forall k \geqslant 2$.

Decomposable arrangements

- For each flat $X \in L(\mathcal{A})$, let $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$.
- The inclusions $\mathcal{A}_X \subset \mathcal{A}$ give rise to maps $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$; get map
$$j\colon M(\mathcal{A}) \longrightarrow \prod_{X \in L_2(\mathcal{A})} M(\mathcal{A}_X)\,.$$
- The induced homomorphism on π_1 yields a morphism
$$\mathfrak{h}(j_\sharp)\colon \mathfrak{h}(G) \longrightarrow \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}(G_X)\,.$$

Theorem (Papadima–S. 2006)

The map $\mathfrak{h}_k(j_\sharp)$ is a surjection for each $k \geqslant 3$ and an iso for $k = 2$.

Definition

$\mathcal{A}$ is *decomposable* if the map $\mathfrak{h}_3(j_\sharp)$ is an isomorphism.

Example

Let $\mathcal{A}(\Gamma) = \{z_i - z_j = 0 : (i,j) \in \mathsf{E}(\Gamma)\} \subset \mathcal{A}_n$ be a graphic arrangement. Then $\mathcal{A}(\Gamma)$ is decomposable if and only if Γ contains no K_4 subgraph.

Theorem (Papadima–S. 2006)

Let $\mathcal{A}$ *be a decomposable arrangement, and let* $G = G(\mathcal{A})$*. Then*

- *The map* $\mathfrak{h}'(j_\sharp)\colon \mathfrak{h}'(G) \to \prod_{X\in L_2(\mathcal{A})} \mathfrak{h}'(G_X)$ *is an isomorphism of graded Lie algebras.*
- *The map* $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ *is an isomorphism*
- *For each* $k \geqslant 2$*, the group* $\operatorname{gr}_k(G)$ *is free abelian of rank* $\phi_k(G) = \sum_{X\in L_2(\mathcal{A})} \phi_k(F_{\mu(X)})$*.*

Theorem (Porter–S. 2020)

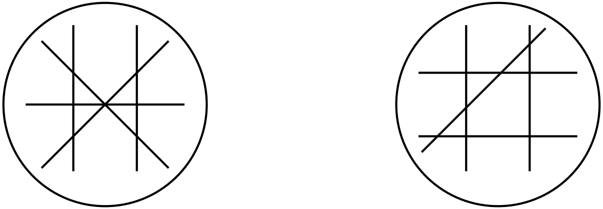
Let $\mathcal{A}$ *and* $\mathcal{B}$ *be decomposable arrangements with* $L_{\leqslant 2}(\mathcal{A}) \cong L_{\leqslant 2}(\mathcal{B})$*. Then, for each* $k \geqslant 2$*,*

$$G(\mathcal{A})/\gamma_k(G(\mathcal{A})) \cong G(\mathcal{B})/\gamma_k(G(\mathcal{B})).$$

Theorem

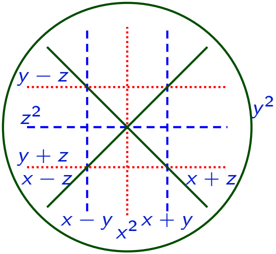
If $\mathcal{A}$ *is decomposable (over* $\mathbb{Q}$*) and* G/G'' *is residually nilpotent, then the monodromy action on* $H_1(F(\mathcal{A});\mathbb{Q})$ *is trivial.*

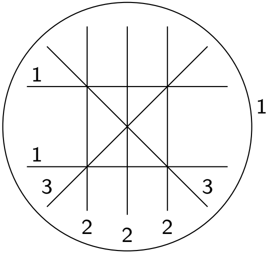
Falk's pair of arrangements



- Both $\mathcal{A}$ and $\hat{\mathcal{A}}$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\hat{\mathcal{A}})$. Nevertheless, $M(\mathcal{A}) \simeq M(\hat{\mathcal{A}})$.
- Both arrangements are decomposable; their Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- Nevertheless, $K = \pi_1(F)$ is *not* isomorphic to $\hat{K} = \pi_1(\hat{F})$. In fact:
  - $K/K'' \not\cong \hat{K}/\hat{K}''$, since $\mathcal{V}_2^1(K) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2^1(\hat{K}) = \{1\}$.
  - $K/\gamma_3(K) \not\cong \hat{K}/\gamma_3(\hat{K})$, since $H_2(K/\gamma_3(K); \mathbb{Z}) = \mathbb{Z}_3$, yet $H_2(\hat{K}/\gamma_3(\hat{K}); \mathbb{Z}) = 0$.

The deleted B_3 arrangement



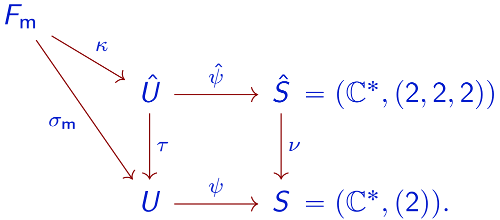


- Let $M = M(\mathcal{A})$, where $\mathcal{A}$ is the deleted B_3 arrangement.
- $\mathcal{V}_1^1(M) \subset (\mathbb{C}^*)^8$ contains local components corresponding to the 6 triple points and 1 quadruple point, 5 components corresponding to braid sub-arrangements, and a component of the form $\rho \cdot T$, where

$$T = \{(t^2, t^{-2}, t^{-1}, t^{-1}, 1, 1, t, t) : t \in \mathbb{C}^*\},$$
$$\rho = (1, 1, -1, -1, -1, -1, 1, 1).$$

- This translated subtorus arises from an orbifold fibration $\psi \colon M \to (\mathbb{C}^*, (2))$.

- Let m be multiplicities as shown above. The monodromy acts trivially on $H_1(F_{\mathrm{m}};\mathbb{Q})$, but not on $H_1(F_{\mathrm{m}};\mathbb{Z})$, which has torsion subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ on which the monodromy acts as $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.
- The pencil ψ lifts to 3-fold cover $\hat{U}$ obtained from pull-back diagram



- This diagram determines all the $12 + 1 + 7 = 20$ components of $\mathcal{V}^1_1(F_{\mathrm{m}})$, as well as those of $\mathcal{V}^1_2(F_{\mathrm{m}})$ and $\mathcal{V}^1_3(F_{\mathrm{m}})$. Moreover,

k	1	2	3	4	5
$\mathrm{gr}_k(\pi_1(F_{\mathrm{m}}))$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^5$	$\mathbb{Z}^{28} \oplus \mathbb{Z}_2^{15}$	$\mathbb{Z}^{78} \oplus \mathbb{Z}_2^{41}$	$\mathbb{Z}^{252} \oplus \mathbb{Z}_2^{117}$
$\mathrm{gr}_k(\pi_1(F_{\mathrm{m}})/\pi_1(F_{\mathrm{m}})'')$	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^5$	$\mathbb{Z}^{28} \oplus \mathbb{Z}_2^{15}$	$\mathbb{Z}^{48} \oplus \mathbb{Z}_2^{?}$	$\mathbb{Z}^{68} \oplus \mathbb{Z}_2^{?}$

Yoshinaga's icosidodecahedral arrangement

- The icosidodecahedron is the convex hull of 30 vertices given by the even permutations of $(0,0,\pm 1)$ and $\frac{1}{2}(\pm 1, \pm\phi, \pm\phi^2)$, where $\phi = (1+\sqrt{5})/2$.
- It gives rise to an arrangement of 16 hyperplanes in $\mathbb{R}^3$, whose complexification is the icosidodecahedral arrangement $\mathcal{A}$ in $\mathbb{C}^3$.
- $H_1(F;\mathbb{Z}) = \mathbb{Z}^{15} \oplus \mathbb{Z}_2$. Thus, the algebraic monodromy of the Milnor fibration is trivial over $\mathbb{Q}$ and $\mathbb{Z}_p$ ($p>2$), but not over $\mathbb{Z}$.
- Hence, $\operatorname{gr}(\pi_1(F)) \cong \operatorname{gr}(\pi_1(U))$, away from the prime 2. Moreover,

k	1	2	3	4
$\operatorname{gr}_k(K)$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$	$\mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$	$\mathbb{Z}^{1,405} \oplus \mathbb{Z}_2^{?}$
$\operatorname{gr}_k(K/K'')$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{45} \oplus \mathbb{Z}_2^7$	$\mathbb{Z}^{250} \oplus \mathbb{Z}_2^{43}$	$\mathbb{Z}^{530} \oplus \mathbb{Z}_2^{?}$

References

- A.I. Suciu, *Milnor fibrations of arrangements with trivial algebraic monodromy*, preprint arxiv:2402.03619

- A.I. Suciu, *On the topology and combinatorics of decomposable arrangements*, preprint.