On the topology and combinatorics of decomposable arrangements

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Associated graded Lie algebras

- The *lower central series* of a group G is defined inductively by $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_{k+1}(G) = [G, \gamma_k(G)]$.
- $[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$, $\forall k, \ell \geqslant 1$.
- In particular, it is a *central* series, i.e., $[G, \gamma_k(G)] \subseteq \gamma_{k+1}(G)$.
- It is also a *normal* series, i.e., $\gamma_k(G) \lhd G$. Each quotient,

$$\operatorname{gr}_k(G) := \gamma_k(G)/\gamma_{k+1}(G)$$

  lies in the center of $G/\gamma_{k+1}(G)$, and thus is an abelian group.
- If G is finitely generated (for short, f.g.), then so are its LCS quotients. Set $\phi_k(G) := \operatorname{rank} \operatorname{gr}_k(G)$.
- $\operatorname{gr}(G) := \bigoplus_{k\geqslant 1} \operatorname{gr}_k(G)$ is a graded Lie algebra (over $\mathbb{Z}$), with addition induced by the group multiplication and with Lie bracket $[\ ,\ ]\colon \operatorname{gr}_k \times \operatorname{gr}_\ell \to \operatorname{gr}_{k+\ell}$ induced by the group commutator.

- The *Chen Lie algebra* of G is defined as $\mathrm{gr}(G/G'')$. We have a surjection $\mathrm{gr}_k(G) \twoheadrightarrow \mathrm{gr}_k(G/G'')$, which is an iso for $k \leqslant 3$.
- Assuming G is f.g., write $\theta_k(G) = \operatorname{rank} \mathrm{gr}_k(G/G'')$ for the *Chen ranks*. We have $\phi_k(G) \geqslant \theta_k(G)$, with equality for $k \leqslant 3$.
- Example: if F_n is the free group of rank n, then
  - $\mathrm{gr}(F_n)$ is the free Lie algebra $\mathrm{Lie}(\mathbb{Z}^n)$.
  - $\mathrm{gr}_k(F_n)$ is free abelian, of rank $\phi_k(F_n) = \frac{1}{k}\sum_{d|k} \mu(d) n^{k/d}$.
  - $\mathrm{gr}_k(F_n/F_n'')$ is free abelian, $\theta_k(F_n) = (k-1)\binom{n+k-2}{k}$, for $k \geqslant 2$.

Holonomy Lie algebra

- For G a f.g. group, define

$$\mathfrak{h}(G) := \mathrm{Lie}(H)/\mathrm{ideal}(\mathrm{im}(\nabla_G)),$$

  where $H = G_{\mathrm{abf}} = G_{\mathrm{ab}}/\,\mathrm{Tors}$, $\mathrm{Lie}(H)_1 = H$, $\mathrm{Lie}(H)_2 = H \wedge H$, and $\nabla_G \colon H^2(G;\mathbb{Z})^\vee \to H \wedge H$ is the dual of the cup-product map $\cup_G \colon H^1(G;\mathbb{Z}) \wedge H^1(G;\mathbb{Z}) \to H^2(G;\mathbb{Z})$.

- $\mathfrak{h}(G)$ is a quadratic Lie algebra, determined solely by $H^{\leqslant 2}(G)$.

- There is a natural epi $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$ which restricts to isos in degrees 1 and 2, and factors through epi $\mathfrak{h}(G)/\mathfrak{h}(G)'' \twoheadrightarrow \mathrm{gr}(G/G'')$.

- G is 1-*formal* if the Malcev Lie algebra $\mathfrak{m}(G) := \mathrm{Prim}(\widehat{\mathbb{Q}[G]})$ is isomorphic to the LCS-completion completion of $\mathfrak{h}(G) \otimes \mathbb{Q}$.

- In that case, $\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$ and $\mathfrak{h}(G)/\mathfrak{h}(G)'' \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G/G'') \otimes \mathbb{Q}$.

The Alexander invariant

- Let $G'' = (G')'$. We then have an extension

$$1 \longrightarrow G'/G'' \longrightarrow G/G'' \longrightarrow G/G' \longrightarrow 1\,.$$

- Both $G/G' = G_{\mathrm{ab}}$ are $G'/G'' = (G')_{\mathrm{ab}}$ are abelian groups, and G/G'' is the maximal metabelian quotient of G.

- The *Alexander invariant* is

$$B(G) := G'/G''.$$

  viewed as a $\mathbb{Z}G_{\mathrm{ab}}$-module via $gG' \cdot xG'' = gxg^{-1}G''$ for $g \in G$ and $x \in G'$.

- If X is a connected CW-complex with $\pi_1(X) = G$, then

$$B(G) = H_1(X^{\mathrm{ab}}, \mathbb{Z}) = H_1(X, \mathbb{Z}[G_{\mathrm{ab}}]),$$

  where $q\colon X^{\mathrm{ab}} \to X$ is the universal abelian cover and G_{ab} acts on $B(G)$ by automorphisms induced by deck transformations.

- A homomorphism $\alpha\colon G \to H$ induces compatible homomorphisms, $\alpha_{\mathrm{ab}}\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ and $B(\alpha)\colon B(G) \to B(H)$.
- That is, if $\tilde{\alpha}_{\mathrm{ab}}\colon \mathbb{Z}[G_{\mathrm{ab}}] \to \mathbb{Z}[H_{\mathrm{ab}}]$ is the linear extension of α_{ab} to a ring map, then $B(\alpha)$ is a morphism of modules covering α_{ab}, i.e., $B(\alpha)(rm) = \tilde{\alpha}_{\mathrm{ab}}(r) \cdot B(\alpha)(m)$ for all $r \in \mathbb{Z}[G_{\mathrm{ab}}]$ and $m \in B(G)$.
- $B(\alpha)$ factors as
$$B(G) \to B(H)_\alpha \to B(H),$$
where $B(H)_\alpha$ is the $\mathbb{Z}[G_{\mathrm{ab}}]$-module obtained from $B(H)$ by restriction of scalars via $\tilde{\alpha}$.

THEOREM (MASSEY 1980)

Let $I = \ker(\varepsilon\colon \mathbb{Z}[G_{\mathrm{ab}}] \to \mathbb{Z})$ *be the augmentation ideal. Then* $I^n B(G) = \gamma_{n+2}(G/G'')$, *and thus* $\operatorname{gr}_n(B) \cong \operatorname{gr}_{n+2}(G/G'')$, *for all* $n \geqslant 0$.

Hence:
$$\operatorname{Hilb}(\operatorname{gr}(B(G) \otimes \mathbb{Q}), t) = \sum_{n \geqslant 0} \theta_{n+2}(G) t^n.$$

Infinitesimal Alexander invariant

- Let G be a f.g. group. The symmetric algebra $\operatorname{Sym}(G_{\mathrm{abf}})$ is isomorphic to $\operatorname{gr}(\mathbb{Z}[G_{\mathrm{abf}}])$.
- If we identify G_{abf} with $\mathbb{Z}^r$, where $r = b_1(G)$, then $\operatorname{Sym}(G_{\mathrm{abf}})$ gets identified with the polynomial ring $\mathbb{Z}[x_1, \dots, x_r]$.
- Now let $\mathfrak{h}(G)$ be the holonomy Lie algebra of G. The *infinitesimal Alexander invariant* of G is the quotient group

$$\mathfrak{B}(G) := \mathfrak{h}(G)'/\mathfrak{h}(G)'',$$

viewed as a graded module over $\operatorname{Sym}(G_{\mathrm{abf}})$. The module structure comes from the exact sequence

$$0 \longrightarrow \mathfrak{h}(G)'/\mathfrak{h}(G)'' \longrightarrow \mathfrak{h}(G)/\mathfrak{h}(G)'' \longrightarrow \mathfrak{h}(G)/\mathfrak{h}(G)' \longrightarrow 0$$

via the adjoint action of $\mathfrak{h}(G)/\mathfrak{h}(G)' = \mathfrak{h}_1(G) = G_{\mathrm{abf}}$ on $\mathfrak{h}(G)'/\mathfrak{h}(G)''$ given by $g \cdot \bar{x} = \overline{[g, x]}$ for $g \in \mathfrak{h}_1(G)$ and $x \in \mathfrak{h}(G)'$, and with the grading inherited from the one on $\mathfrak{h}(G)$.

- When G admits a finite, commutator-relators presentation, $\mathfrak{B}(G)$ is isomorphic to the "linearization" of $B(G)$.
- The holonomy Chen ranks of a f.g. group G are defined as $\bar\theta_n(G) = \dim_{\mathbb{Q}} (\mathfrak{h}(G) \otimes \mathbb{Q}/\mathfrak{h}''(G) \otimes \mathbb{Q})_n$. Then $\theta_n(G) \leqslant \bar\theta_n(G)$ and

$$\bar\theta_n(G) = \dim_{\Bbbk} \mathfrak{B}_{n-2}(G) \otimes \mathbb{Q}, \text{ for all } n \geqslant 2.$$

Theorem

Let G be a 1-formal group. There is then a natural, filtration-preserving isomorphism of completed modules, $B\widehat{(G) \otimes} \mathbb{Q} \cong \mathfrak{B}\widehat{(G) \otimes} \mathbb{Q}$.

Corollary

If G is 1-formal, then $\mathrm{gr}(B(G) \otimes \mathbb{Q}) \cong \mathfrak{B}(G) \otimes \mathbb{Q}$, as graded modules over the ring $\mathrm{gr}(\mathbb{Q}[G_{\mathrm{ab}}]) \cong \mathrm{Sym}(H_1(G;\mathbb{Q}))$.

Corollary

If G is 1-formal, then $\theta_n(G) = \bar\theta_n(G) = \dim_{\mathbb{Q}} \mathfrak{B}_{n-2}(G) \otimes \mathbb{Q}$ for all $n \geqslant 2$.

Characteristic varieties

- Let G be a f.g. group. The character group, $\mathbb{T}_G = \operatorname{Hom}(G, \mathbb{C}^*)$, is an abelian, complex algebraic group, with identity 1 the trivial representation.

- The coordinate ring of $\mathbb{T}_G$ is the group algebra $\mathbb{C}[G_{\mathrm{ab}}]$; thus, we may identify $\mathbb{T}_G$ with $\operatorname{maxSpec}(\mathbb{C}[G_{\mathrm{ab}}])$.

- Since each character $\rho\colon G \to \mathbb{C}^*$ factors through G_{ab}, the map $\mathrm{ab}\colon G \twoheadrightarrow G_{\mathrm{ab}}$ induces an isomorphism, $\mathrm{ab}^*\colon \mathbb{T}_{G_{\mathrm{ab}}} \xrightarrow{\simeq} \mathbb{T}_G$.

- Let X be a connected CW-complex with finite 1-skeleton and with $\pi_1(X) = G$. Identify $\rho \in \mathbb{T}_G$ with a rank one local system $\mathbb{C}_\rho$ on X. For each $k \geqslant 1$, the *depth k characteristic variety* of G is defined as

$$\mathcal{V}_k(G) := \{\rho \in \mathbb{T}_G \mid \dim_{\mathbb{C}} H_1(X, \mathbb{C}_\rho) \geqslant k\}.$$

- The sets $\mathcal{V}_k(G)$ do not depend on the choice of a space X as above.

Theorem

$\mathcal{V}_k(G) = \operatorname{supp}\left(\bigwedge^k B(G) \otimes \mathbb{C}\right)$, *at least away from* $1 \in \mathbb{T}_G$.

Example

Let F_n be a free group of rank $n \geqslant 2$. Then
$\mathcal{V}_1(F_n) = \cdots = \mathcal{V}_{n-1}(F_n) = (\mathbb{C}^*)^n$ and $\mathcal{V}_n(F_n) = \{1\}$.

Example

Let Σ_g be a Riemann surface of genus $g \geqslant 2$. Then
$\mathcal{V}_1(\pi_1(\Sigma_g)) = \cdots = \mathcal{V}_{2g-2}(\pi_1(\Sigma_g)) = (\mathbb{C}^*)^{2g}$ and $\mathcal{V}_{2g-1}(\pi_1(\Sigma_g)) = \{1\}$.

Example

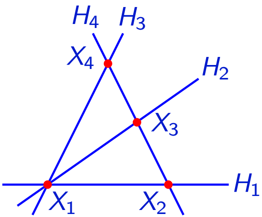
Let K be a knot in S^3, and let $G = \pi_1(S^3 \backslash K)$. Since $G_{\text{ab}} = \mathbb{Z}$, we may identify $\mathbb{T}_G = \mathbb{C}^*$. The variety $\mathcal{V}_1(G)$ consists of 1, together with the roots of the Alexander polynomial of the knot, $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$.

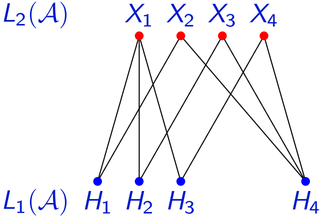
Resonance varieties

- Let $H^* = H^*(G;\mathbb{C})$ be the cohomology algebra of a f.g. group G.
- For each element $a \in H^1$, we have $a^2 = -a^2$, and so $a^2 = 0$. Thus, left-multiplication by a defines a cochain complex,
$$(H, \delta_a)\colon \; H^0 \xrightarrow{\delta_a^0} H^1 \xrightarrow{\delta_a^1} H^2,$$
with differentials $\delta_a^i(u) = a \cdot u$ for $u \in H^i$.
- For each $k \geqslant 1$, the *depth* k *resonance variety* of G is defined as
$$\mathcal{R}_k(G) := \{a \in H^1 \mid \dim_{\mathbb{C}} H^1(H, \delta_a) \geqslant k\}.$$
- These sets are homogeneous algebraic subvarieties of the affine space $H^1 = H^1(G;\mathbb{C})$.
- $\mathcal{R}_k(G) = \operatorname{supp}\left(\bigwedge^k \mathfrak{B}(G) \otimes \mathbb{C}\right)$ for all $k \geqslant 1$, at least away from $0 \in H^1(G;\mathbb{C})$.
- If G is a 1-formal group, then $\mathrm{TC}_1(\mathcal{V}_k(G)) = \mathcal{R}_k(G)$, for all $k \geqslant 1$.

Hyperplane arrangements

- An *arrangement of hyperplanes* is a finite collection $\mathcal{A}$ of codimension 1 linear (or affine) subspaces in $\mathbb{C}^d$.
- For each $H \in \mathcal{A}$, let f_H be a linear form with $\ker(f_H) = H$; set $f = \prod_{H\in\mathcal{A}} f_H$.
- *Intersection lattice* $L(\mathcal{A})$: poset of all intersections of $\mathcal{A}$, ordered by reverse inclusion, and ranked by codimension.





- *Complement* $M(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H\in\mathcal{A}} H$: a smooth algebraic variety and a Stein manifold homotopic to a finite, connected CW-complex of dim d.

Example (The Boolean arrangement)

- $\mathcal{B}_n$: all coordinate hyperplanes $z_i = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{B}_n)$: Boolean lattice of subsets of $\{0,1\}^n$.
- $M(\mathcal{B}_n)$: complex algebraic torus $(\mathbb{C}^*)^n \simeq K(\mathbb{Z}^n, 1)$.

Example (The braid arrangement)

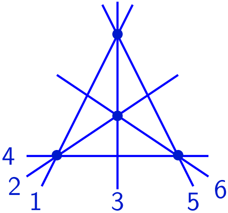
- $\mathcal{A}_n$: all diagonal hyperplanes $z_i - z_j = 0$ in $\mathbb{C}^n$.
- $L(\mathcal{A}_n)$: lattice of partitions of $[n] := \{1, \dots, n\}$, ordered by refinement.
- $M(\mathcal{A}_n)$: (ordered) configuration space of n distinct points in $\mathbb{C}$; it is a classifying space $K(P_n, 1)$ for the pure braid group on n strands, P_n.

- The space $M = M(\mathcal{A})$ admits a minimal cell structure.
- The groups $H_q(M;\mathbb{Z})$ are finitely generated and torsion-free, with ranks given by $\sum_{q=0}^{\ell} b_q(M)t^q = \sum_{X\in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{rank}(X)}$, where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is defined by $\mu(\mathbb{C}^d) = 1$ and $\mu(X) = -\sum_{Y \supsetneqq X} \mu(Y)$.

Cohomology rings of arrangements

- Let E be the $\mathbb{Z}$-exterior algebra on degree 1 cohomology classes $e_H = \frac{1}{2\pi i}[d\log(f_H)]$ dual to the meridians x_H around $H \in \mathcal{A}$.
- Let $\partial\colon E^* \to E^{*-1}$ be the differential given by $\partial(e_H) = 1$, and set $e_X = \prod_{H \supseteq X} e_H$ for each $X \in \mathcal{L}(A)$.
- Arnold, Brieskorn, Orlik–Solomon showed: $H^*(M;\mathbb{Z}) \cong E/I$, where $I = \langle \partial e_X : \operatorname{rank}(X) < |X| \rangle$.
- M is $\mathbb{Q}$-formal (albeit not $\mathbb{Z}_p$-formal, in general).

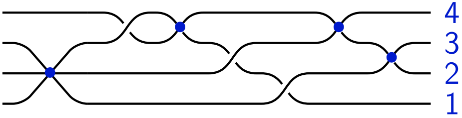
Example



- $E = \bigwedge(e_1, \dots, e_6)$
- $I = \langle (e_1 - e_4)(e_2 - e_4), (e_1 - e_5)(e_3 - e_5), (e_2 - e_6)(e_3 - e_6), (e_4 - e_6)(e_5 - e_6) \rangle$

Fundamental groups of arrangements

- Let $\mathcal{A}' = \{H \cap \mathbb{C}^2\}_{H \in \mathcal{A}}$ be a generic planar section of $\mathcal{A}$. Then the arrangement group, $G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$, is isomorphic to $\pi_1(M(\mathcal{A}'))$.
- So let $\mathcal{A}$ be an arrangement of n affine lines in $\mathbb{C}^2$. Taking a generic projection $\mathbb{C}^2 \to \mathbb{C}$ yields the braid monodromy $\alpha = (\alpha_1, \dots, \alpha_s)$, where $s = \#\{\text{multiple points}\}$ and the braids $\alpha_r \in P_n \hookrightarrow \text{Aut}(F_n)$ can be read off an associated braided wiring diagram,



- The group $G(\mathcal{A})$ has a presentation with meridional generators $x_1, \dots, x_n$ and commutator relators $x_i \alpha_j(x_i)^{-1}$.

Lie algebras of arrangement groups

- (Kohno 1983) The holonomy Lie algebra of $G = G(\mathcal{A})$ is determined by $L_{\leqslant 2}(\mathcal{A})$:

$$\mathfrak{h}(G) = \mathrm{Lie}(x_H : H \in \mathcal{A}) \Big/ \mathrm{ideal} \left\{ \Big[x_H, \sum_{\substack{K\in\mathcal{A} \\ K\supset Y}} x_K \Big] \ : \ \begin{matrix} H\in\mathcal{A}, Y\in L_2(\mathcal{A}) \\ H\supset Y \end{matrix} \right\}.$$

- Since M is formal, G is 1-formal. Hence, $\mathrm{gr}(G) \otimes \mathbb{Q} \cong \mathfrak{h}(G) \otimes \mathbb{Q}$ is determined by $H^{\leqslant 2}(M;\mathbb{Q})$, and thus, by $L_{\leqslant 2}(\mathcal{A})$.
- $U(\mathfrak{h}(G) \otimes \mathbb{Q}) = \mathrm{Ext}^1_A(\mathbb{Q},\mathbb{Q}) = \overline{A}^!$, the quadratic dual of the quadratic closure of the OS-algebra $A = H^*(M,\mathbb{Q})$.
- (Papadima–S. 2004) $L_{\leqslant 2}(\mathcal{A})$ also determines the Chen ranks $\theta_k(G)$.
- Explicit combinatorial formulas for the LCS ranks $\phi_k(G)$ are known in some cases, but not in general.

- (Falk–Randell 1985) If $\mathcal{A}$ is *supersolvable* with exponents $d_1, \dots, d_\ell$, then $\phi_k(G) = \sum_{i=1}^{\ell} \phi_k(F_{d_i})$. (Also follows from Koszulity of $H^*(M, \mathbb{Q})$ and Koszul duality.)
- (Porter–S. 2020) The map $\mathfrak{h}_3(G) \to \operatorname{gr}_3(G)$ is an isomorphism, but it is not known whether $\mathfrak{h}_3(G)$ is torsion-free.
- (S. 2002) The groups $\operatorname{gr}_k(G)$ may have non-zero torsion for $k \gg 0$. E.g., if $G = G(\text{MacLane})$, then $\operatorname{gr}_5(G) = \mathbb{Z}^{87} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_3$.
- (S. 2002): Is the torsion in $\operatorname{gr}(G)$ combinatorially determined?
- (Artal Bartolo, Guerville-Ballé, and Viu-Sos 2020): Answer: No!
- There are two arrangements of 13 lines, $\mathcal{A}^{\pm}$, each one with 11 triple points and 2 quintuple points, such that $\operatorname{gr}_k(G^+) \cong \operatorname{gr}_k(G^-)$ for $k \leqslant 3$, yet $\operatorname{gr}_4(G^+) = \mathbb{Z}^{211} \oplus \mathbb{Z}_2$ and $\operatorname{gr}_4(G^-) = \mathbb{Z}^{211}$.

Nilpotent quotients of arrangement groups

- The quotient $G/\gamma_3(G)$ is determined by $L_{\leqslant 2}(\mathcal{A})$. Indeed, in the central extension,

$$0 \longrightarrow \operatorname{gr}_2(G) \longrightarrow G/\gamma_3(G) \longrightarrow G_{ab} \longrightarrow 0,$$

  we have $\operatorname{gr}_2(G) = (I^2)^\vee$ and the k-invariant $H_2(G_{ab}) \to \operatorname{gr}_2(G)$ is dual of the inclusion $I^2 \hookrightarrow E^2 = \bigwedge^2 G_{ab}$.

- (G. Rybnikov 1994): $G/\gamma_4(G)$ is not always determined by $L_{\leqslant 2}(\mathcal{A})$.

- There are two arrangements of 13 lines, $\mathcal{A}^\pm$, each one with 15 triple points, such that $L(\mathcal{A}^+) \cong L(\mathcal{A}^-)$, and therefore $G^+/\gamma_3(G^+) \cong G^-/\gamma_3(G^-)$ and $\operatorname{gr}_3(G^+) \cong \operatorname{gr}_3(G^-)$, but $G^+/\gamma_4(G^+) \not\cong G^-/\gamma_4(G^-)$.

- The difference can be explained in terms of (generalized) Massey triple products over $\mathbb{Z}_3$.

Cohomology jump loci of arrangements

- Let $\mathcal{A}$ be an arrangement of n hyperplanes, and $M = M(\mathcal{A})$. Then $\mathcal{R}^1_1(M)$ is a (finite) union of linear subspaces in $H^1(M, \mathbb{C}) \cong \mathbb{C}^n$.
- Each subspace L has dimension at least 2, and each pair of subspaces meets transversely at 0.
- $\mathcal{R}^1_k(M)$ is the union of those linear subspaces L that have dimension at least $k+1$.
- Each component of $\mathcal{R}^1_1(M)$ corresponds to a *multinet* on a sub-arrangement of $\mathcal{A}$.
- The characteristic variety $\mathcal{V}^1_1(M)$ is a finite union of torsion-translates of algebraic subtori of $\mathrm{Hom}(\pi_1(M), \mathbb{C}^*) = (\mathbb{C}^*)^n$.
- All components of $\mathcal{V}^1_1(M)$ passing through $1 \in (\mathbb{C}^*)^n$ are of the form $T = \exp(L)$, for some component $L \subset \mathcal{R}^1_1(M)$.
- In general, though, there are translated subtori in $\mathcal{V}^1_1(M)$, which are not *a priori* determined by $L(\mathcal{A})$.

Localized sub-arrangements

- For each flat $X \in L(\mathcal{A})$, let $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$ be the *localization* of $\mathcal{A}$ at X.
- Choosing a point x_0 close to $0 \in \mathbb{C}^d$, we can make x_0 a common basepoint for both $M(\mathcal{A})$ and all the local complements $M(\mathcal{A}_X)$.
- Let $j_X \colon M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$ be the inclusion corresponding to $\mathcal{A}_X \subset \mathcal{A}$.
- There exist basepoint-preserving maps $r_X \colon M(\mathcal{A}_X) \to M(\mathcal{A})$ such that $j_X \circ r_X \simeq \mathrm{id}$ relative to x_0.
- Hence, the induced homomorphism $(j_X)_\sharp \colon G(\mathcal{A}) \to G(\mathcal{A}_X)$ is surjective and $(r_X)_\sharp \colon G(\mathcal{A}_X) \to G(\mathcal{A})$ is injective.
- When X is a 2-flat, $\mathcal{A}_X$ is a pencil of $|X| = \mu(X) + 1$ hyperplanes. Hence, $M(\mathcal{A}_X) \cong (\mathbb{C} \backslash \{\mu(X) \text{ points}\}) \times \mathbb{C}^* \times \mathbb{C}^{d-2}$, and so $M(\mathcal{A}_X)$ is a classifying space for the group $G(\mathcal{A}_X) = F_{\mu(X)} \times \mathbb{Z}$.

Decomposing the holonomy Lie algebra

- The maps $j_X\colon M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$ give rise to a map
$$j\colon M(\mathcal{A}) \longrightarrow \prod_{X\in L_2(\mathcal{A})} M(\mathcal{A}_X)\,.$$
- The homomorphism induced by j on fundamental groups,
$$j_\sharp\colon G(\mathcal{A}) \longrightarrow \prod_{X\in L_2(\mathcal{A})} G(\mathcal{A}_X) =: G(\mathcal{A})^{\mathrm{loc}}$$
yields a morphism between the respective holonomy Lie algebras,
$$\mathfrak{h}(j_\sharp)\colon \mathfrak{h}(G) \longrightarrow \prod_{X\in L_2(\mathcal{A})} \mathfrak{h}(G_X) =: \mathfrak{h}(G)^{\mathrm{loc}}.$$

Theorem (Papadima–S. 2006)

The map $\mathfrak{h}_k(j_\sharp)$ is a surjection for each $k \geqslant 3$ and an iso for $k = 2$.

Definition

We say $\mathcal{A}$ is *decomposable* if the map $\mathfrak{h}_3(j_\sharp)$ is an isomorphism. Likewise, $\mathcal{A}$ is *decomposable over* $\mathbb{Q}$ if the map $\mathfrak{h}_3(j_\sharp) \otimes \mathbb{Q}$ is an isomorphism.

- That is to say, $\mathcal{A}$ is decomposable if $\mathfrak{h}_3(\mathcal{A})$ is free abelian of rank as small as possible, namely,

$$\operatorname{rank} \mathfrak{h}_3(G)^{\mathrm{loc}} = \sum_{X \in L_2(\mathcal{A}_X)} \binom{\mu(X)}{2}.$$

- Question: are decomposability and $\mathbb{Q}$-decomposability equivalent?
- If $\mathcal{A}$ is decomposable, and $\mathcal{B} \subset \mathcal{A}$, then $\mathcal{B}$ is decomposable.
- Let $\mathcal{A}(\Gamma) = \{z_i - z_j = 0 : (i,j) \in \mathsf{E}(\Gamma)\}$ be a graphic arrangement. Then $\mathcal{A}(\Gamma)$ is decomposable if and only if Γ contains no K_4 subgraph.

Theorem (Papadima–S. 2006)

Let $\mathcal{A}$ be a decomposable arrangement, with group $G = G(\mathcal{A})$. Then:

- $\mathfrak{h}'(j_\sharp)\colon \mathfrak{h}'(G) \to \mathfrak{h}'(G)^{\mathrm{loc}}$ *is an isomorphism of graded Lie algebras.*
- *The map $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ is an isomorphism.*
- *For each $k \geqslant 2$, the group $\operatorname{gr}_k(G)$ is free abelian of rank*

$$\phi_k(G) = \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)}).$$

Theorem (Papadima–S. 2006)

Let $\mathcal{A}$ be a decomposable arrangement, with group $G = G(\mathcal{A})$. Then:

- $\operatorname{gr}(G/G'') = \mathfrak{h}(G)/\mathfrak{h}''(G)$, *as graded Lie algebras over* $\mathbb{Z}$.
- $\operatorname{gr}(G/G'')$ *is torsion-free, as a graded abelian group.*
- *The Chen ranks of G, for $k \geqslant 2$, are given by*

$$\theta_k(G) = \sum_{X \in L_2(\mathcal{A})} \theta_k(F_{\mu(X)}).$$

Theorem (Porter–S. 2020)

Let $\mathcal{A}$ and $\mathcal{B}$ be decomposable arrangements with $L_{\leqslant 2}(\mathcal{A}) \cong L_{\leqslant 2}(\mathcal{B})$. Then, for each $k \geqslant 2$,

$$G(\mathcal{A})/\gamma_k(G(\mathcal{A})) \cong G(\mathcal{B})/\gamma_k(G(\mathcal{B})).$$

Alexander invariants of arrangements

- The Alexander invariant of an arrangement $\mathcal{A}$ is defined as

$$B(\mathcal{A}) := B(G(\mathcal{A})) = G'/G'',$$

viewed as a module over the group ring $R = \mathbb{Z}[G_{ab}] = \mathbb{Z}[H_1(M;\mathbb{Z})]$.

- For each flat $X \in L_2(\mathcal{A})$, we also have a "local" Alexander invariant, $B(\mathcal{A}_X)$, viewed as a module over the group ring $R_X = \mathbb{Z}[H_1(M_X;\mathbb{Z})]$.

- The homomorphism $j_\sharp^X\colon G(\mathcal{A}) \to G(\mathcal{A}_X)$ induces a morphism $B(j_\sharp^X)\colon B(\mathcal{A}) \to B(\mathcal{A}_X)$, which covers the ring map $\tilde{j}_*^X\colon R \to R_X$ induced by $j_*^X\colon H_1(M;\mathbb{Z}) \twoheadrightarrow H_1(M_X;\mathbb{Z})$.

- We then obtain an R-morphism, $\Pi\colon B(\mathcal{A}) \to B(\mathcal{A})^{\text{loc}}$, where

$$B(\mathcal{A})^{\text{loc}} := \bigoplus_{X\in L_2(\mathcal{A})} B(\mathcal{A}_X)_{\tilde{j}_*^X}$$

is the R-module obtained from $\bigoplus_X B(\mathcal{A}_X)$ by restriction of scalars.

Decomposable Alexander invariants

Proposition

The R-morphism $\Pi\colon B(\mathcal{A}) \to B(\mathcal{A})^{\mathrm{loc}}$ is surjective.

Definition

We say that the *Alexander invariant of $\mathcal{A}$ decomposes* if the map $\Pi\colon B(\mathcal{A}) \to B(\mathcal{A})^{\mathrm{loc}}$ is an isomorphism of R-modules.

(A similar definition works over $\mathbb{Q}$.)

- Let $I = \ker(\varepsilon\colon \mathbb{Z}[G_{\mathrm{ab}}] \to \mathbb{Z})$ be the augmentation ideal, and let $\widehat{B}$ be the completion of B in the I-adic topology.
- The R-module $B = B(\mathcal{A})$ is separated if $\bigcap_{k\geqslant 1} I^k B = \{0\}$, or, equivalently, the map $B \to \widehat{B}$ is injective.
- If $G(\mathcal{A})$ is residually nilpotent, then $B(\mathcal{A})$ is separated.

- We define the *infinitesimal Alexander invariant of $\mathcal{A}$* as

$$\mathfrak{B}(\mathcal{A}) := \mathfrak{B}(G(\mathcal{A})) = \mathfrak{h}'(\mathcal{A})/\mathfrak{h}''(\mathcal{A}),$$

  viewed as a module over the symmetric algebra $S = \operatorname{Sym}[G_{\mathrm{ab}}]$.
- Since $G_{\mathrm{ab}} = H_1(M(\mathcal{A}); \mathbb{Z})$ and $R = \mathbb{Z}[H_1(M(\mathcal{A}); \mathbb{Z})]$, the ring S is isomorphic (as a graded ring) to $\operatorname{gr}(R)$.
- To each $X \in L_2(\mathcal{A})$ there corresponds $\mathfrak{B}(\mathcal{A}_X)$, a module over $S_X = \operatorname{Sym}[H_1(M(\mathcal{A}_X); \mathbb{Z})] \cong \operatorname{gr}(R_X)$.
- As before, we obtain a surjective morphism of graded S-modules, $\bar{\Pi}\colon \mathfrak{B}(\mathcal{A}) \to \mathfrak{B}(\mathcal{A})^{\mathrm{loc}}$.

Definition

We say that the *infinitesimal Alexander invariant of $\mathcal{A}$ decomposes* if the map $\bar{\Pi}\colon \mathfrak{B}(\mathcal{A}) \to \mathfrak{B}(\mathcal{A})^{\mathrm{loc}}$ is an isomorphism of S-modules.

Theorem

- *If* $\mathcal{A}$ *is decomposable, then* $\mathfrak{B}(\mathcal{A})$ *is decomposable.*
- *If* $\mathcal{A}$ *is* $\mathbb{Q}$*-decomposable, then* $\mathfrak{B}(\mathcal{A})$ *and* $\widehat{B(\mathcal{A})}$ *are* $\mathbb{Q}$*-decomposable.*
- *If* $\mathcal{A}$ *is* $\mathbb{Q}$*-decomposable and* $B(\mathcal{A}) \otimes \mathbb{Q}$ *is separated, then* $B(\mathcal{A})$ *is* $\mathbb{Q}$*-decomposable.*

Corollary

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ *and set* $\widetilde{L}_2(\mathcal{A}) := \{X \in L_2(\mathcal{A}) : \mu(X) > 1\}$.

- *If* $\mathcal{A}$ *is* $\mathbb{Q}$*-decomposable, then* $\mathcal{R}^1_1(M(\mathcal{A})) = \bigcup_{X \in \widetilde{L}_2(\mathcal{A})} L_X$*, where*

$$L_X = \Big\{ x \in \mathbb{C}^n : \sum_{H_i \in \mathcal{A}_X} x_i = 0 \text{ and } x_i = 0 \text{ if } H_i \notin \mathcal{A}_X \Big\}.$$

- *If* $\mathcal{A}$ *is* $\mathbb{Q}$*-decomposable and* $B(\mathcal{A}) \otimes \mathbb{Q}$ *is separated, then* $\mathcal{V}^1_1(M(\mathcal{A})) = \bigcup_{X \in \widetilde{L}_2(\mathcal{A})} T_X$*, where*

$$T_X = \Big\{ t \in (\mathbb{C}^*)^n : \prod_{H_i \in \mathcal{A}_X} t_i = 1 \text{ and } t_i = 1 \text{ if } H_i \notin \mathcal{A}_X \Big\}.$$

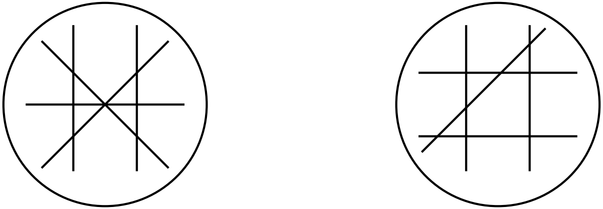
Milnor fibrations

- Let $\mathcal{A}$ be a central arrangement of n hyperplanes in $\mathbb{C}^d$, with defining polynomial $f = \prod_{H\in\mathcal{A}} f_H$.
- The polynomial map $f\colon \mathbb{C}^d \to \mathbb{C}$ restricts to a smooth fibration, $f\colon M \to \mathbb{C}^*$, called the *Milnor fibration* of $\mathcal{A}$.
- The *Milnor fiber* is $F := f^{-1}(1)$. The monodromy of the fibration, $h\colon F \to F$, is given by $h(z) = e^{2\pi i/n} z$.

Theorem

If $\mathcal{A}$ is decomposable over $\mathbb{Q}$ and $B(\mathcal{A}) \otimes \mathbb{Q}$ is separated, then the algebraic monodromy, $h_*\colon H_1(F;\mathbb{Q}) \to H_1(F;\mathbb{Q})$ is the identity, and therefore, $b_1(F) = n - 1$.

Falk's pair of arrangements



- Both $\mathcal{A}$ and $\hat{\mathcal{A}}$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\hat{\mathcal{A}})$. Nevertheless, $M(\mathcal{A}) \simeq M(\hat{\mathcal{A}})$.
- Both arrangements are decomposable, and their Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- Nevertheless, $K = \pi_1(F)$ is *not* isomorphic to $\hat{K} = \pi_1(\hat{F})$. In fact:
  - $K/K'' \not\cong \hat{K}/\hat{K}''$, since $\mathcal{V}_2^1(K) \cong \mathbb{Z}_3$, yet $\mathcal{V}_2^1(\hat{K}) = \{1\}$.
  - $K/\gamma_3(K) \not\cong \hat{K}/\gamma_3(\hat{K})$, since $H_2(K/\gamma_3(K); \mathbb{Z}) = \mathbb{Z}_3$, yet $H_2(\hat{K}/\gamma_3(\hat{K}); \mathbb{Z}) = 0$.

References

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