Artin kernels and Milnor fibrations of arrangements

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Workshop on Artin groups and arrangements: topology, geometry, and combinatorics

SLMath, Berkeley

March 15, 2024

RAAGs and arrangements: A comparison

Simplicial complex $L \subset 2^{[n]}$	Toric complex $T_L \subset T^n$ $\#k$-cells $= b_k$, $\dim T_L = \dim L + 1$
Graph $\Gamma = (V, E) = L^{(1)}$	RAAG $G_\Gamma = \pi_1(T_L) =$ $\langle v \in V : [v, w] = 1 \text{ if } \{v, w\} \in E \rangle$
Flag complex Δ_Γ	Classifying space $K(G_\Gamma, 1) = T_{\Delta_\Gamma}$ Hence, G_Γ is torsion-free
Cohomology ring $H^*(T_L; \Bbbk)$	Exterior Stanley–Reisner ring $\Bbbk\langle L \rangle = \bigwedge(e_v)/(e_\sigma : \sigma \notin L)$
L flag complex	$\Bbbk\langle L \rangle$ Koszul algebra
T_L formal over $\mathbb{Q}$ and $\mathbb{Z}$	$A_{\mathrm{PL}}(T_L) \simeq_{\mathrm{CDGA}} H^*(X; \mathbb{Q})$ $C^*(T_L; \mathbb{Z}) \simeq_{\mathrm{DGA}} H^*(X; \mathbb{Z})$ Hence, G_Γ is 1-formal

Arrangement $\mathcal{A}$ in $\mathbb{C}^d$	Lattice $L(\mathcal{A}) = \{\bigcap_{H\in\mathcal{B}} H : \mathcal{B} \subset \mathcal{A}\}$
Defining polynomial $f = \prod_{H\in\mathcal{A}} f_H$, $\ker(f_H) = H$	Complement $M(\mathcal{A}) = \mathbb{C}^d \setminus \bigcup_{H\in\mathcal{A}} H$ Minimal cell structure, $\dim \leqslant d$
$G(\mathcal{A}) = \pi_1(M(\mathcal{A}))$, gens $\{x_H\}_{H\in\mathcal{A}}$, commutator rels	No finite $K(G(\mathcal{A}),1)$ in general Is $G(\mathcal{A})$ torsion-free?
Cohomology $H^*(M(\mathcal{A});\mathbb{Z})$ is torsion-free	Orlik–Solomon algebra $A = \bigwedge(e_H)/(\partial e_X : \operatorname{codim}(X) < \lvert X\rvert)$
$b_k(M) = \sum_{X\in L_k(\mathcal{A})} (-1)^k \mu(X)$	$\mu \colon L(\mathcal{A}) \to \mathbb{Z}$ defined by $\mu(\mathbb{C}^d) = 1$ and $\mu(X) = -\sum_{Y \supsetneq X} \mu(Y)$
$\mathcal{A}$ supersolvable $\Longrightarrow$	A Koszul algebra
$M(\mathcal{A})$ formal over $\mathbb{Q}$	$H^*(X;\mathbb{R}) \to \Omega_{\mathrm{dR}}(M(\mathcal{A}))$ q-iso $H^k(X;\mathbb{C})$ has pure MHS, type (k,k) Hence, $G(\mathcal{A})$ is 1-formal
$M(\mathcal{A})$ not formal over $\mathbb{Z}_p$	Massey products in $H^2(G(\mathcal{A});\mathbb{Z}_p)$

Artin kernels and Milnor fibers: A comparison

Weights m_v on vertices of $\Gamma = L^{(1)}$	Homomorphism $\chi\colon G_\Gamma \to \mathbb{Z}$, $\chi(v) = m_v$ Surjective if $\gcd(m_v) = 1$
Cover $T_L^\chi \to T_L$	Artin kernel $N_\chi = \pi_1(T_L^\chi) = \ker(\chi)$
$m_v = 1$ for all $v \in V$	Bestvina–Brady group N_Γ
Γ connected	N_Γ finitely generated
Δ_Γ simply connected	N_Γ finitely presented
If $H_i(T_L^\chi;\Bbbk)$ $\Bbbk\mathbb{Z}$-trivial, for $i \leqslant r$: $H^{\leqslant r}(T_L^\chi;\Bbbk) = H^{\leqslant r}(T_L;\Bbbk)/(\chi_\Bbbk)$	

Weights $\mathfrak{m}_H$ on $H \in \mathcal{A}$	Polynomial $f_{\mathrm{m}} = \prod_{H\in\mathcal{A}} f_H^{m_H}$
Milnor fibration	$f_{\mathrm{m}} \colon M(\mathcal{A}) \to \mathbb{C}^*$
Milnor fiber	$F_{\mathrm{m}} = f_{\mathrm{m}}^{-1}(1)$; $F(\mathcal{A}) = f^{-1}(1)$
Monodromy ($N = \sum_{H\in\mathcal{A}} m_H$)	$h \colon F_{\mathrm{m}} \to F_{\mathrm{m}}$, $h(z) = e^{2\pi i/N} z$
F_{m} finite complex, $\dim \leqslant d-1$	F_{m} connected if $\gcd(m_H) = 1$
$\chi \colon G(\mathcal{A}) \twoheadrightarrow \mathbb{Z}$, $\chi(x_H) = m_H$	$F_{\mathrm{m}} \simeq M^{\chi}$, $\pi_1(F_{\mathrm{m}}) = \ker(\chi)$ $F_{\mathrm{m}} \to \mathbb{P}(M)$ a regular $\mathbb{Z}_N$-cover
$H_*(F_{\mathrm{m}};\mathbb{Z})$ may have torsion	Even $H_1(F(\mathcal{A});\mathbb{Z})$ may have torsion
$F(\mathcal{A})$ not always 1-formal	$H^1(F(\mathcal{A});\mathbb{C})$ may be non-pure

Theorem

Let $\mathcal{A}$ be an arrangement of lines in $\mathbb{C}^2$, with group $G = G(\mathcal{A})$. The following are equivalent:

1. *G is a right-angled Artin group.*
2. *G is a finite direct product of finitely generated free groups.*
3. *The multiplicity graph of $\mathcal{A}$ is a forest.*

- There exist graphs Γ such that the Bestvina–Brady group N_Γ is finitely presented, yet not isomorphic to either an Artin group $G_{\Gamma'}$, or an arrangement group $G(\mathcal{A})$.
- There exist arrangements $\mathcal{A}$ such that $G(\mathcal{A}) \cong N_\Gamma$ for some graph Γ, but $G(\mathcal{A}) \not\cong G_{\Gamma'}$ for any graph Γ'.

Associated graded and Chen Lie algebras

Lower central series	$\gamma_1(G) = G$, $\gamma_2(G) = G'$ $\gamma_{k+1}(G) = [G, \gamma_k(G)]$
It is a normal, central series	$[\gamma_k(G), \gamma_\ell(G)] \subseteq \gamma_{k+\ell}(G)$
Associated graded Lie algebra	$\operatorname{gr}(G) = \bigoplus_{k\geqslant 1} \gamma_k(G)/\gamma_{k+1}(G)$
Chen Lie algebra	$\operatorname{gr}(G/G'')$
If G finitely generated, then $\operatorname{gr}_k(G)$ f.g. (abelian) groups	LCS ranks: $\phi_k(G) = \operatorname{rank} \operatorname{gr}_k(G)$ Chen ranks: $\theta_k(G) = \operatorname{rank} \operatorname{gr}_k(G/G'')$
$\operatorname{gr}_k(G) \twoheadrightarrow \operatorname{gr}_k(G/G'')$ iso for $k \leqslant 3$	$\phi_k(G) \geqslant \theta_k(G)$ with $=$ for $k \leqslant 3$

Holonomy and Malcev Lie algebras

G f.g., $H = G_{\mathrm{abf}} = G_{\mathrm{ab}}/\,\mathrm{Tors}$	$\nabla_G = \cup_G^{\vee}\colon H^2(G;\mathbb{Z})^{\vee} \to H \wedge H$
Holonomy Lie algebra	$\mathfrak{h}(G) := \mathrm{Lie}(H)/\mathrm{ideal}(\mathrm{im}(\nabla_G))$
This is a quadratic Lie algebra	$\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$ $\mathfrak{h}(G)/\mathfrak{h}(G)'' \twoheadrightarrow \mathrm{gr}(G/G'')$
Malcev Lie algebra	$\mathfrak{m}(G) := \mathrm{Prim}(\widehat{\mathbb{Q}[G]})$ $\mathrm{gr}(\mathfrak{m}(G)) \cong \mathrm{gr}(G) \otimes \mathbb{Q}$
G is 1-formal if $\Longrightarrow$ $\mathfrak{m}(G) \cong \widehat{\mathfrak{h}(G)} \otimes \mathbb{Q}$	$\mathfrak{h}(G) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G) \otimes \mathbb{Q}$ $\mathfrak{h}(G)/\mathfrak{h}(G)'' \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}(G/G'') \otimes \mathbb{Q}$

Lie algebras of RAAGs

$\mathfrak{h}(G_\Gamma) = \mathrm{Lie}(V)/([v,w]=0)$ if $\{v,w\} \in E$	$\mathfrak{h}(G_\Gamma) \xrightarrow{\simeq} \mathrm{gr}(G_\Gamma)$ $\mathrm{gr}(G_\Gamma)$ torsion-free, ranks given by
$P_\Gamma(t) = \sum_{n\geqslant 0} f_n(\Gamma) t^n$ $f_n(\Gamma) = \#\{n\text{-cliques in } \Gamma\}$	$\prod_{k=1}^{\infty} (1-t^k)^{\phi_k} = P_\Gamma(-t)$
$\mathfrak{h}_\Gamma/\mathfrak{h}_\Gamma'' \xrightarrow{\simeq} \mathrm{gr}(G_\Gamma/G_\Gamma'')$	$\mathrm{gr}(G_\Gamma/G_\Gamma'')$ torsion-free, ranks given by $\sum_{k=2}^{\infty} \theta_k t^k = Q_\Gamma\big(t/(1-t)\big)$
where $Q_\Gamma(t) = \sum_{j\geqslant 2} c_j(\Gamma) t^j$ and $c_j(\Gamma) = \sum_{W\subset V:\, \lvert W\rvert = j} \tilde{b}_0(\Gamma_W)$	
(Γ, ℓ) labeled graph $G_{\Gamma,\ell}$ Artin group	$\Gamma_{\mathrm{odd}} = (\mathrm{V}, \mathrm{E}')$, $\mathrm{E}' = \{e : \ell(e) \text{ odd}\}$ $\widetilde{\Gamma} = (\widetilde{\mathrm{V}}, \widetilde{\mathrm{E}})$: $\widetilde{\mathrm{V}}$ = components of Γ_{odd} $\widetilde{\mathrm{E}}$ induced edges from E'
$\mathfrak{m}(G_{\Gamma,\ell}) \cong \mathfrak{m}(G_{\widetilde{\Gamma}})$	$\phi_k(G_{\Gamma,\ell}) = \phi_k(G_{\widetilde{\Gamma}})$ $\theta_k(G_{\Gamma,\ell}) = \theta_k(G_{\widetilde{\Gamma}})$

Lie algebras of hyperplane arrangements

<table>
<tr><td colspan="2">$\mathfrak{h}(G) = \mathrm{Lie}(x_H : H \in \mathcal{A}) \Big/ \mathrm{ideal} \Big\{ \Big[x_H, \sum_{\substack{K \in \mathcal{A} \\ K \supset Y}} x_K \Big] : H \in \mathcal{A}, Y \in L_2(\mathcal{A}), H \supset Y \Big\}$</td></tr>
<tr><td>$\mathrm{gr}(G) \otimes \mathbb{Q} \cong \mathfrak{h}(G) \otimes \mathbb{Q}$
$\phi_k(G)$ determined by $L_{\leqslant 2}(\mathcal{A})$</td><td>$\mathrm{gr}(G/G'') \otimes \mathbb{Q} \cong \mathfrak{h}(G)/\mathfrak{h}(G)'' \otimes \mathbb{Q}$
$\theta_k(G)$ determined by $L_{\leqslant 2}(\mathcal{A})$</td></tr>
<tr><td colspan="2">$\mathcal{A}$ supersolvable $\implies \prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = \mathrm{Poin}(M(\mathcal{A}), -t)$</td></tr>
<tr><td colspan="2">$\mathcal{A}$ decomposable $\implies \displaystyle\prod_{k=1}^{\infty} (1 - t^k)^{\phi_k} = (1 - t)^{\lvert \mathcal{A} \rvert - \sum_{X \in L_2(\mathcal{A})} \mu(X)} \prod_{X \in L_2(\mathcal{A})} (1 - \mu(X) t)$</td></tr>
<tr><td>$\mathfrak{h}_3(G) \xrightarrow{\simeq} \mathrm{gr}_3(G)$</td><td>Question: Is $\mathfrak{h}_3(G)$ torsion-free?</td></tr>
<tr><td>$\mathrm{gr}_k(G)$ may have non-zero torsion for $k \gg 0$</td><td>Question: Is the torsion in $\mathrm{gr}(G)$ combinatorially determined? Answer: No.</td></tr>
<tr><td colspan="2">$\exists\ \mathcal{A}^{\pm}$ with $L(\mathcal{A}^+) \cong L(\mathcal{A}^-)$, yet $\mathrm{tors}(\mathrm{gr}_4(G^+)) \not\cong \mathrm{tors}(\mathrm{gr}_4(G^-))$</td></tr>
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Alexander invariants

Alexander invariant	$B(G) := G'/G''$ as $\mathbb{Z}G_{\mathrm{ab}}$-module via $gG' \cdot xG'' = gxg^{-1}G''$ $(g \in G,\ x \in G')$
If $\pi_1(X) = G$ and $X^{\mathrm{ab}} \to X$	$B(G) = H_1(X^{\mathrm{ab}}, \mathbb{Z}) = H_1(X, \mathbb{Z}[G_{\mathrm{ab}}])$
Let $I = \ker(\varepsilon\colon \mathbb{Z}[G_{\mathrm{ab}}] \to \mathbb{Z})$	$I^k B(G) = \gamma_{k+2}(G/G'')$, and so $\theta_k(G) = \operatorname{rank} \operatorname{gr}_{k-2}(B(G))$, $\forall k \geqslant 2$
Infinitesimal Alexander invariant	$\mathfrak{B}(G) := \mathfrak{h}(G)'/\mathfrak{h}(G)''$, as module over $\operatorname{Sym}(G_{\mathrm{abf}}) = \operatorname{gr}(\mathbb{Z}[G_{\mathrm{abf}}])$.

Theorem

Let G be a 1-formal group. Then,

(1) $\widehat{B(G)} \otimes \mathbb{Q} \cong \widehat{\mathfrak{B}(G)} \otimes \mathbb{Q}$.

(2) $\operatorname{gr}(B(G)) \otimes \mathbb{Q} \cong \mathfrak{B}(G) \otimes \mathbb{Q}$.

(3) $\theta_k(G) = \dim_{\mathbb{Q}} \mathfrak{B}_{k-2}(G) \otimes \mathbb{Q}$ *for* $k \geqslant 2$.

Cohomology jump loci

Character group of $G = \pi_1(X)$	$\mathbb{T}_G := \operatorname{Hom}(G, \mathbb{C}^*) = H^1(X; \mathbb{C}^*)$ $\mathbb{T}_G \cong \mathbb{T}_G^0 \times \operatorname{tors}(G_{\mathrm{ab}})$, $\mathbb{T}_G^0 \cong (\mathbb{C}^*)^{b_1(G)}$
Characteristic varieties	$\mathcal{V}^i_s(X) := \{\rho \in \mathbb{T}_G : \dim H_i(X; \mathbb{C}_\rho) \geqslant s\}$ $\mathcal{W}^i_s(X) := \mathcal{V}^i_s(X) \cap \mathbb{T}^0_G$
$\mathcal{V}_s(G) := \mathcal{V}^1_s(X)$ depend only on G/G''	$\mathcal{V}_s(G) = \operatorname{supp}\left(\bigwedge^s B(G) \otimes \mathbb{C}\right)$ away from $1 \in \mathbb{T}_G$
$A = H^*(X; \mathbb{C})$, $a \in A^1$ $\rightsquigarrow$	$(A, \cdot a)\colon A^0 \xrightarrow{\cdot a} A^1 \xrightarrow{\cdot a} A^2 \longrightarrow \cdots$
Resonance varieties	$\mathcal{R}^i_s(X) := \{a \in A^1 : \dim H^i(A, \cdot a) \geqslant s\}$
$\mathcal{R}_s(G) := \mathcal{R}^1_s(X)$	$\mathcal{R}_s(G) = \operatorname{supp}\left(\bigwedge^s \mathfrak{B}(G) \otimes \mathbb{C}\right)$ away from $0 \in A^1$
If X is a k-formal:	$\operatorname{TC}_1(\mathcal{V}^i_s(X)) = \mathcal{R}^i_s(X)$ for $i \leqslant k$

The resonance varieties $\mathcal{R}^i_s(M)$ of $M = M(\mathcal{A})$ are finite unions of linear subspaces in $\mathbb{C}^{\lvert\mathcal{A}\rvert}$	The characteristic varieties $\mathcal{V}^i_s(M)$ are finite unions of torsion-translated subtori of $(\mathbb{C}^*)^{\lvert\mathcal{A}\rvert}$
The components of $\mathcal{R}^1_1(M)$ correspond to *multinets* on sub-arrangements of $\mathcal{A}$. Each subspace has dimension at least 2, and each pair of subspaces meets transversely at 0.	
M is an abelian duality space of dimension $r = \operatorname{rank}(\mathcal{A})$: $H^*(X, \mathbb{Z}G_{\mathrm{ab}})$ concentrated in deg r	The jump loci of $\mathcal{A}$ propagate: $\mathcal{R}^1_1(M) \subseteq \cdots \subseteq \mathcal{R}^r_1(M)$ $\mathcal{V}^1_1(M) \subseteq \cdots \subseteq \mathcal{V}^r_1(M)$

Cohomology jump loci of RAAGs

The resonance varieties of T_L are unions of coordinate subspaces inside $H_V := H^1(T_L;\mathbb{C})$	$\mathcal{R}^i_s(T_L) = \bigcup_{\substack{W\subseteq V \\ \exists\sigma\in L_{V\setminus W},\ \dim \widetilde{H}_{i-1-\lvert\sigma\rvert}(\mathrm{lk}_{L_W}(\sigma),\mathbb{C})\geqslant s}} H_W$
For a RAAG G_Γ:	$\mathcal{R}^1_1(G_\Gamma) = \bigcup_{\substack{W\subseteq V \\ \Gamma_W \text{ disconnected}}} H_W$
The characteristic varieties of T_L and G_Γ are unions of coordinate subtori $\mathbb{T}_W \subset T_V$, on same indexing sets	
T_L is an abelian duality space $\Longleftrightarrow$ L is Cohen–Macaulay	L CM $\Longrightarrow$ resonance varieties of L propagate Question: Is the converse true?

Almost direct products

Theorem (Falk–Randell 1985/88)

Let $G = K \rtimes_{\varphi} Q$. If Q acts trivially on K_{ab}, then

- $\mathrm{gr}(G) = \mathrm{gr}(K) \rtimes_{\tilde{\varphi}} \mathrm{gr}(Q)$, *where* $\tilde{\varphi}\colon \mathrm{gr}(Q) \to \mathrm{Der}(\mathrm{gr}(K))$.
- *If K and Q are residually nilpotent, then G is residually nilpotent.*

Theorem

Let $G = K \rtimes_{\varphi} Q$. If Q acts trivially on $K_{\mathrm{abf}} := K_{\mathrm{ab}}/\mathrm{Tors}$, then

- $\mathrm{gr}(G) \otimes \mathbb{Q} \cong (\mathrm{gr}(K) \rtimes_{\tilde{\varphi}} \mathrm{gr}(Q)) \otimes \mathbb{Q}$.
- *If K and Q are RTFN, then G is RTFN.*

If K_{abf} is f.g., Q is torsion-free abelian, and Q acts trivially on $H_1(K;\mathbb{Q})$, then

- $\mathrm{gr}_{\geqslant 2}(K) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G) \otimes \mathbb{Q}$, *and so* $\phi_k(K) = \phi_k(G)$ *for* $k \geqslant 2$.
- $\mathrm{gr}_{\geqslant 2}(K/K'') \otimes \mathbb{Q} \xrightarrow{\simeq} \mathrm{gr}_{\geqslant 2}(G/G'') \otimes \mathbb{Q}$, *and so* $\theta_k(K) = \theta_k(G)$ *for* $k \geqslant 2$.

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be an exact sequence of f.g. groups.*

- *If* Q *is abelian and acts trivially on* K_{ab}, *then* $\iota^*\colon \mathbb{T}_G \to \mathbb{T}_K$ *restricts to maps* $\iota^*\colon \mathcal{V}^1_s(G) \to \mathcal{V}^1_s(K)$ *for all* $s \geqslant 1$; *furthermore,* $\iota^*\colon \mathcal{V}^1_1(G) \to \mathcal{V}^1_1(K)$ *is a surjection.*
- *If* Q *is torsion-free abelian and acts trivially on* $H_1(K;\mathbb{Q})$, *then* $\iota^*\colon \mathbb{T}^0_G \twoheadrightarrow \mathbb{T}^0_K$ *restricts to maps* $\iota^*\colon \mathcal{W}^1_s(G) \to \mathcal{W}^1_s(K)$ *for all* $s \geqslant 1$; *furthermore,* $\iota^*\colon \mathcal{W}^1_1(G) \to \mathcal{W}^1_1(K)$ *is a surjection.*

Theorem

Let $1 \to K \xrightarrow{\iota} G \to Q \to 1$ *be an exact sequence of f.g. groups. Suppose* G *and* K *are* 1*-formal,* Q *is torsion-free abelian, and* Q *acts trivially on* $H_1(K;\mathbb{Q})$. *Then* $\iota^*\colon H^1(G;\mathbb{C}) \twoheadrightarrow H^1(K;\mathbb{C})$ *restricts to maps* $\iota^*\colon \mathcal{R}^1_s(G) \to \mathcal{R}^1_s(K)$ *for all* $s \geqslant 1$; *furthermore,* $\iota^*\colon \mathcal{R}^1_1(G) \to \mathcal{R}^1_1(K)$ *is surjective.*

Bestvina–Brady groups

Theorem (Papadima–S. 2007/2009, S. 2021)

Suppose $\Gamma = (\mathsf{V}, \mathsf{E})$ *is connected. Then*

- *In the split exact sequence* $1 \to N_\Gamma \xrightarrow{\iota} G_\Gamma \xrightarrow{\pi} \mathbb{Z} \to 1$, *the group* $\mathbb{Z}$ *acts trivially on* $(N_\Gamma)_{\mathrm{ab}}$.
- $\operatorname{gr}_{\geqslant 2}(N_\Gamma) \cong \operatorname{gr}_{\geqslant 2}(G_\Gamma)$. *and* $\operatorname{gr}_{\geqslant 2}(N_\Gamma/N_\Gamma'') \cong \operatorname{gr}_{\geqslant 2}(G_\Gamma/G_\Gamma'')$.
- $\phi_k(N_\Gamma) = \phi_k(G_\Gamma)$ *and* $\theta_k(N_\Gamma) = \theta_k(G_\Gamma)$ *for all* $k \geqslant 2$.
- *If* $\kappa(\Gamma) = 1$, *then* $\mathcal{V}^1_1(N_\Gamma) = \operatorname{Hom}(N_\Gamma, \mathbb{C}^*)$ *and* $\mathcal{R}^1_1(N_\Gamma) = H^1(N_\Gamma; \mathbb{C})$.
- *If* $\kappa(\Gamma) > 1$, *then the irreducible components of* $\mathcal{V}^1_1(N_\Gamma)$, *respectively* $\mathcal{R}^1_1(N_\Gamma)$, *are the subtori* $\mathbb{T}'_{\mathsf{W}} = \iota^*(\mathbb{T}_{\mathsf{W}})$, *respectively the subspaces* $H'_{\mathsf{W}} = \iota^*(H_{\mathsf{W}})$, *of dimension* $|\mathsf{W}|$, *one for each subset* $\mathsf{W} \subset \mathsf{V}$, *maximal among those for which the induced subgraph* Γ_{W} *is disconnected.*

THEOREM

Let $(\mathcal{A}, \mathrm{m})$ *be a multi-arrangement, and let* F_{m} *be its Milnor fiber. Suppose* $h_*\colon H_1(F_{\mathrm{m}}; \mathbb{Z}) \to H_1(F_{\mathrm{m}}; \mathbb{Z})$ *is the identity. Then*

- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}})) \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M))$.
- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}})/\pi_1(F_{\mathrm{m}})'') \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M)/\pi_1(M)'')$.

THEOREM

Suppose $h_*\colon H_1(F_{\mathrm{m}}; \mathbb{Q}) \to H_1(F_{\mathrm{m}}; \mathbb{Q})$ *is the identity. Then*

- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}})) \otimes \mathbb{Q} \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M)) \otimes \mathbb{Q}$.
- $\mathrm{gr}_{\geqslant 2}(\pi_1(F_{\mathrm{m}})/\pi_1(F''_{\mathrm{m}})) \otimes \mathbb{Q} \cong \mathrm{gr}_{\geqslant 2}(\pi_1(M)/\pi_1(M)'') \otimes \mathbb{Q}$.

Hence, $\phi_k(\pi_1(F_{\mathrm{m}})) = \phi_k(\pi_1(M))$ *and* $\theta_k(\pi_1(F_{\mathrm{m}})) = \theta_k(\pi_1(M))$, $\forall k \geqslant 2$.

Theorem

Let $\sigma_{\mathrm{m}}\colon F_{\mathrm{m}} \to U = \mathbb{P}(M)$ *be the restriction of* $M \to \mathbb{P}(M)$. *Suppose the monodromy* $h\colon F_{\mathrm{m}} \to F_{\mathrm{m}}$ *induces the identity on* $H_1(F_{\mathrm{m}};\mathbb{Q})$. *Then,*

- *The induced homomorphism* $\sigma_{\mathrm{m}}^*\colon H^1(U;\mathbb{C}) \to H^1(F_{\mathrm{m}};\mathbb{C})$ *is an isomorphism that identifies* $\mathcal{R}^1_s(U)$ *with* $\mathcal{R}^1_s(F_{\mathrm{m}})$, *for all* $s \geqslant 1$.
- *The induced homomorphism* $\sigma_{\mathrm{m}}^*\colon H^1(U;\mathbb{C}^*) \to H^1(F_{\mathrm{m}};\mathbb{C}^*)^0$ *is a surjection with kernel isomorphic to* $\mathbb{Z}_N$. *Moreover,*
  - *For each* $s \geqslant 1$, *the map* σ_{m}^* *establishes a bijection between the sets of irreducible components of* $\mathcal{V}^1_s(U)$ *and* $\mathcal{W}^1_s(F_{\mathrm{m}})$ *that pass through the identity.*
  - *The map* $\sigma_{\mathrm{m}}^*\colon \mathcal{V}^1_1(U) \to \mathcal{W}^1_1(F_{\mathrm{m}})$ *is a surjection.*

Alexander invariants of arrangements

- Alexander invariant: $B(\mathcal{A}) := B(G(\mathcal{A})) = G'/G''$, viewed as a module over $R = \mathbb{Z}[G_{\text{ab}}] = \mathbb{Z}[H_1(M;\mathbb{Z})]$.
- [Cohen–S. 1999] The homomorphisms $j^X_\sharp \colon G(\mathcal{A}) \to G(\mathcal{A}_X)$ induce a surjective R-morphism, $\Pi\colon B(\mathcal{A}) \to B(\mathcal{A})^{\text{loc}} := \bigoplus_{X\in L_2(\mathcal{A})} B(\mathcal{A}_X)$.
- Infinitesimal Alexander invariant: $\mathfrak{B}(\mathcal{A}) := \mathfrak{B}(G(\mathcal{A})) = \mathfrak{h}'(\mathcal{A})/\mathfrak{h}''(\mathcal{A})$, viewed as a module over $S = \text{Sym}[G_{\text{ab}}] \cong \text{gr}(R)$.
- There is an epimorphism of graded S-modules, $\bar{\Pi}\colon \mathfrak{B}(\mathcal{A}) \to \mathfrak{B}(\mathcal{A})^{\text{loc}}$.
- Hence, the Chen ranks of $\mathcal{A}$ admit the lower bound

$$\theta_k(G(\mathcal{A})) \geqslant (k-1) \sum_{X\in L_2(\mathcal{A})} \binom{\mu(X)+k-2}{k},$$

valid for all $k \geqslant 2$, with equality for $k = 2$.

Decomposable arrangements

- For each flat $X \in L(\mathcal{A})$, let $\mathcal{A}_X := \{H \in \mathcal{A} \mid H \supset X\}$.
- The inclusions $\mathcal{A}_X \subset \mathcal{A}$ give rise to maps $M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_X)$; get map

$$j \colon M(\mathcal{A}) \longrightarrow \prod_{X \in L_2(\mathcal{A})} M(\mathcal{A}_X) .$$

- The induced homomorphism on π_1 yields a morphism

$$\mathfrak{h}(j_\sharp) \colon \mathfrak{h}(G) \longrightarrow \prod_{X \in L_2(\mathcal{A})} \mathfrak{h}(G_X) =: \mathfrak{h}(G)^{\mathrm{loc}}.$$

Theorem (Papadima–S. 2006)

The map $\mathfrak{h}_k(j_\sharp)$ *is a surjection for each* $k \geqslant 3$ *and an iso for* $k = 2$.

Definition

$\mathcal{A}$ is *decomposable* if the map $\mathfrak{h}_3(j_\sharp)$ is an isomorphism; that is, $\mathfrak{h}_3(G)$ is free abelian of rank as small as possible, namely, $\sum_{X \in L_2(\mathcal{A}_X)} \binom{\mu(X)}{2}$.

- A similar definition works over $\mathbb{Q}$ (or any field $\Bbbk$).
- Question: are decomposability and $\mathbb{Q}$-decomposability equivalent?
- If $\mathcal{A}$ is decomposable, and $\mathcal{B} \subset \mathcal{A}$, then $\mathcal{B}$ is decomposable.
- Let $\mathcal{A}(\Gamma) = \{z_i - z_j = 0 : (i,j) \in \mathsf{E}(\Gamma)\}$ be a graphic arrangement. Then $\mathcal{A}(\Gamma)$ is decomposable if and only if Γ contains no K_4 subgraph.

Theorem (Papadima–S. 2006)

Let $\mathcal{A}$ be a decomposable arrangement, with group $G = G(\mathcal{A})$. Then:

- $\mathfrak{h}'(j_\sharp)\colon \mathfrak{h}'(G) \to \mathfrak{h}'(G)^{\mathrm{loc}}$ *is an isomorphism of graded Lie algebras.*
- *The map* $\mathfrak{h}(G) \twoheadrightarrow \operatorname{gr}(G)$ *is an isomorphism.*
- *For each* $k \geqslant 2$, *the group* $\operatorname{gr}_k(G)$ *is free abelian of rank*

$$\phi_k(G) = \sum_{X \in L_2(\mathcal{A})} \phi_k(F_{\mu(X)}).$$

Theorem (Papadima–S. 2006)

Let $\mathcal{A}$ be a decomposable arrangement, with group $G = G(\mathcal{A})$. Then:

- $\operatorname{gr}(G/G'') = \mathfrak{h}(G)/\mathfrak{h}''(G)$*, as graded Lie algebras over* $\mathbb{Z}$.
- $\operatorname{gr}(G/G'')$ *is torsion-free, as a graded abelian group.*
- *The Chen ranks of* G*, for* $k \geqslant 2$*, are given by*

$$\theta_k(G) = \sum_{X \in L_2(\mathcal{A})} \theta_k(F_{\mu(X)}).$$

Theorem (Porter–S. 2020)

Let $\mathcal{A}$ and $\mathcal{B}$ be decomposable arrangements with $L_{\leqslant 2}(\mathcal{A}) \cong L_{\leqslant 2}(\mathcal{B})$. Then, for each $k \geqslant 2$,

$$G(\mathcal{A})/\gamma_k(G(\mathcal{A})) \cong G(\mathcal{B})/\gamma_k(G(\mathcal{B})).$$

- The *Alexander invariant of* $\mathcal{A}$ *decomposes* if the map $\Pi\colon B(\mathcal{A}) \to B(\mathcal{A})^{\mathrm{loc}}$ is an isomorphism. (Similarly over $\mathbb{Q}$.)
- The *infinitesimal Alexander invariant of* $\mathcal{A}$ *decomposes* if the map $\bar{\Pi}\colon \mathfrak{B}(\mathcal{A}) \to \mathfrak{B}(\mathcal{A})^{\mathrm{loc}}$ is an isomorphism. (Similarly over $\mathbb{Q}$.)
- The R-module $B = B(\mathcal{A})$ is separated if $\bigcap_{k\geqslant 1} I^k B = \{0\}$, or, equivalently, the map $B \to \widehat{B}$ is injective.
- If $G(\mathcal{A})$ is residually nilpotent, then $B(\mathcal{A})$ is separated.

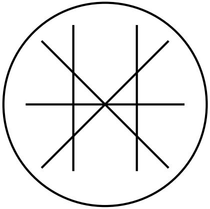
Theorem

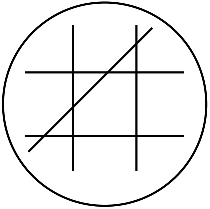
- *If* $\mathcal{A}$ *is decomposable, then* $\mathfrak{B}(\mathcal{A})$ *is decomposable.*
- *If* $\mathcal{A}$ *is* $\mathbb{Q}$*-decomposable, then* $\mathfrak{B}(\mathcal{A})$ *and* $\widehat{B(\mathcal{A})}$ *are* $\mathbb{Q}$*-decomposable.*
- *If* $\mathcal{A}$ *is* $\mathbb{Q}$*-decomposable and* $B(\mathcal{A}) \otimes \mathbb{Q}$ *is separated, then* $B(\mathcal{A})$ *is* $\mathbb{Q}$*-decomposable.*

Corollary

If $\mathcal{A}$ *is* $\mathbb{Q}$*-decomposable and* $B(\mathcal{A}) \otimes \mathbb{Q}$ *is separated, then the monodromy action on* $H_1(F(\mathcal{A});\mathbb{Q})$ *is trivial.*

Falk's pair of arrangements





- Both $\mathcal{A}$ and $\hat{\mathcal{A}}$ have 2 triple points and 9 double points, yet $L(\mathcal{A}) \not\cong L(\hat{\mathcal{A}})$. Nevertheless, $M(\mathcal{A}) \simeq M(\hat{\mathcal{A}})$.
- Both arrangements are decomposable, and their Milnor fibrations have trivial $\mathbb{Z}$-monodromy.
- Nevertheless, $K = \pi_1(F)$ is *not* isomorphic to $\hat{K} = \pi_1(\hat{F})$. In fact:
  - $K/K'' \not\cong \hat{K}/\hat{K}''$, since $\mathcal{V}^1_2(K) \cong \mathbb{Z}_3$, yet $\mathcal{V}^1_2(\hat{K}) = \{1\}$.
  - $K/\gamma_3(K) \not\cong \hat{K}/\gamma_3(\hat{K})$, since $H_2(K/\gamma_3(K); \mathbb{Z}) = \mathbb{Z}_3$, yet $H_2(\hat{K}/\gamma_3(\hat{K}); \mathbb{Z}) = 0$.

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