Name:

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MATH 3175 **Group Theory** **Spring 2024**

Midterm Exam

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1. Let G be an abelian group with identity e, and let H be the set of all elements $a \in G$ that satisfy the equation $a^2 = e$. Prove that H is a subgroup of G.

2. Let $G = \langle a \rangle$ be a group generated by an element a of order 20.

(i) Find all elements of G which generate G.

(ii) List all the elements in the subgroup $\langle a^5 \rangle$, together with their respective orders.

(iii) What are the generators of the subgroup $\langle a^5 \rangle$?

(iv) Find an element in G that has order 4. Does this element generate G?

3. Let $G = \mathrm{GL}(2,2)$ be the group of all invertible 2×2 matrices with entries in $\mathbb{Z}_2$, with group operation given my matrix multiplication.

(i) List all the elements of G and find their orders.

(ii) Does G contain a subgroup of order 3? Why, or why not?

(iii) Is G an abelian group? Why, or why not?

4. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ be the multiplicative group of positive real numbers. Consider the map $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ given by $f(x) = \sqrt{x}$.

(i) Show that f is an homomorphism.

(ii) What is the kernel of f?

(iii) What is the image of f? For each $y \in \operatorname{im}(f)$ find an $x \in \mathbb{R}^+$ such that $f(x) = y$.

(iv) Show that f is an isomorphism, and find the inverse isomorphism.

5. List *all* the homomorphisms from the cyclic group of order 4 to itself. For each such homomorphism, $f\colon \mathbb{Z}_4 \to \mathbb{Z}_4$, indicate what the image of f and the kernel of f are (that is, list the elements of $\operatorname{im}(f)$ and $\ker(f)$).

6. Let $\mathbb{Z}_n^\times$ be the group of units in the ring $\mathbb{Z}_n$, let Q_8 be the quaternion group of order 8, let D_n be the dihedral group of order $2n$, and let S_n be the group of permutations of $\{1, \ldots, n\}$. Show that the following pairs of groups are *not* isomorphic. In each case, explain why.

(i) $\mathbb{Z}_{15}^\times$ and $\mathbb{Z}_8$.

(ii) Q_8 and D_4.

(iii) $Q_8 \times \mathbb{Z}_3$ and $S_3 \times \mathbb{Z}_4$.

7. Let G be a finite group, H a subgroup of G, and K a subgroup of H.

(i) Show that $[G : K] = [G : H] \cdot [H : K]$.

(ii) Suppose $|K| = 10$ and $|G| = 240$. What are the possible values for $|H|$?

8. Let $D_3 = \langle a, b \mid a^3 = b^2 = 1, ba = a^{-1}b \rangle$ be the dihedral group of order 6.

(i) Let $H = \langle a \rangle$ be the cyclic subgroup generated by a. Write down all the right cosets and all the left cosets of H in D_3. Is H a normal subgroup?

(ii) Let $K = \langle b \rangle$ be the cyclic subgroup generated by b. Write down all the right cosets and all the left cosets of K in D_3. Is K a normal subgroup?