**Prof. Alexandru Suciu**

**MATH 3175** **Group Theory** **Spring 2024**

# Homework 6

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**1.** Let $G$ be a group acting on a set $S$, and let $\phi\colon G \to \mathrm{Sym}(S)$ be the homomorphism defined by $\phi(a) = \lambda_a$ for $a \in G$, where $\lambda_a\colon S \to S$ is the bijection given by $\lambda_a(x) = a * x$ for $x \in S$. For each $x \in S$, let $G_x = \{a \in G : a * x = x\}$ be the stabilizer subgroup of $x$. Show that the kernel of $\phi$ coincides with the intersection of all the stabilizer subgroups; that is,

$$\ker(\phi) = \bigcap_{x \in S} G_x \,.$$

**2.** Let $G$ be a group, and let $H \leq G$ be a non-trivial subgroup. Consider the action of the group $H$ on the set $G$ given by left-multiplication; that is, $h * g = hg$ for $h \in H$ and $g \in G$.

(i) What are the orbits of this action?

(ii) What are the stabilizer subgroups of this action?

(iii) What is the subset of $G$ left fixed by this action?

**3.** Let $G = \mathrm{GL}_2(\mathbb{Z}_2)$ be the (multiplicative) group of invertible $2 \times 2$ matrices with entries in $\mathbb{Z}_2$. Let $S = \mathbb{Z}_2 \times \mathbb{Z}_2$, viewed as the set of $2 \times 1$ vectors with entries in $\mathbb{Z}_2$. Consider the action of $G$ on $S$ given by matrix multiplication on vectors; that is, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \quad \text{and} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in S,$$

then

$$A * \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix}.$$

(i) For each element $\vec{v} \in S$, determine the orbit $G\vec{v}$ and stabilizer $G_{\vec{v}}$.

(ii) Determine the set $S^G$ (the subset of $S$ fixed by $G$).

**4.** Let the symmetric group $G = S_5$ act on itself by conjugation. Consider the permutation $\sigma = (123)(45) \in S_5$.

(i) What is the size of the orbit of $\sigma$?

(ii) What are the orders of the elements in this orbit?

(iii) What is the size of the stabilizer of $\sigma$?

**5.** Let $G$ be a group acting on a set $S$.

(i) Suppose $G$ has size 21 and $S$ has size 8. Show that $S^G \neq \emptyset$.

(ii) Give an example where $|G| = 8$, $|S| = 8$, and $S^G = \emptyset$.