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MATH 3175 **Group Theory** **Spring 2024**

Solutions to Homework 5

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1. Let G_1 and G_2 be two groups, with identities e_1 and e_2, respectively. Let $G = G_1 \times G_2$ and let $H = \{(g_1, g_2) \in G_1 \times G_2 : g_2 = e_2\}$. Show that

(i) H is a normal subgroup of G.

First proof. First we check H is a subgroup. Let (g_1, e_2) and (h_1, e_2) be two arbitrary elements in H. Then, since G_1 is a group, we have

$$(g_1, e_2) \cdot (h_1, e_2)^{-1} = (g_1, e_2) \cdot (h_1^{-1}, e_2) = (g_1 h_1^{-1}, e_2) \in H,$$

and this shows $H \leq G$. Next, we show that H is a *normal* subgroup. Let (g_1, g_2) and (h_1, e_2) be arbitrary elements in G and H, respectively. Then

$$\begin{aligned}(g_1, g_2) \cdot (h_1, e_2) \cdot (g_1, g_2)^{-1} &= (g_1, g_2) \cdot (h_1, e_2) \cdot (g_1^{-1}, g_2^{-1}) \\ &= (g_1 h_1 g_1^{-1}, g_2 e_2 g_2^{-1}) \\ &= (g_1 h_1 g_1^{-1}, e_2) \in H,\end{aligned}$$

and this shows $H \triangleleft G$.

Second proof. As shown in part (iii), we have $H = \ker(\varphi)$, where $\varphi\colon G \to G_2$, is the homomorphism given by $\varphi(g_1, g_2) = g_2$. Therefore, $H \triangleleft G$.

(ii) $H \cong G_1$.

Define a map $f\colon H \to G_1$ by setting $f(g_1, e_2) = g_1$ for all $g_1 \in G_1$. This is a homomorphism, since $f((g_1, e_2) \cdot (h_1, e_2)) = f(g_1 h_1, e_2) = g_1 h_1 = f(g_1, e_2) \cdot f(h_1, e_2)$, and also a bijection, with inverse $f^{-1}\colon G_1 \to H$, $f^{-1}(g_1) = (g_1, e_2)$. Hence, f is an isomorphism, and so $H \cong G_1$.

(iii) $G/H \cong G_2$.

Define a map $\varphi\colon G \to G_2$ by setting $\varphi(g_1, g_2) = g_2$. Then φ is a homomorphism, since $\varphi((g_1, g_2) \cdot (h_1, h_2)) = \varphi(g_1 h_1, g_2 h_2) = g_2 h_2 = \varphi(g_1, g_2) \cdot \varphi(h_1, h_2)$.

Clearly, $\operatorname{im}(\varphi) = G_2$, and

$$\ker(\varphi) = \{(g_1, g_2) \in G : \varphi(g_1, g_2) = e_2\} = \{(g_1, g_2) \in G : g_2 = e_2\} = H$$

Thus, by the First Isomorphism Theorem, $G/\ker(\varphi) \cong \operatorname{im}(\varphi)$, and so, $G/H \cong G_2$.

2. Let H be a subgroup of G, and define its *normalizer* as $N(H) := \{g \in G : gHg^{-1} = H\}$.

(i) Show that $N(H)$ is a subgroup of G.

Let $g_1, g_2 \in N(H)$, so that $g_1 H g_1^{-1} = H$ and $g_2 H g_2^{-1} = H$. Then

$$(g_1 g_2) H (g_1 g_2)^{-1} = g_1 (g_2 H g_2^{-1}) g_1^{-1} = g_1 H g_1^{-1} = H \implies g_1 g_2 \in N(H)$$

$$g_1^{-1} (g_1 H g_1^{-1}) g_1 = g_1^{-1} H g_1 \implies H = g_1^{-1} H g_1 \implies g_1^{-1} \in N(H)$$

Thus, $N(H) \leq G$

(ii) Show that the subgroups of G that are conjugate to H are in one-to-one correspondence with the left cosets of $N(H)$ in G.

Let S be the set of subgroups of G that are conjugate to H and let T be the set of left cosets of $N(H)$ in G; that is,

$$S = \{gHg^{-1} : g \in G\} \quad \text{and} \quad T = \{gN(H) : g \in G\}.$$

Define a map $\varphi\colon S \to T$ by setting $\varphi(gHg^{-1}) = gN(H)$. Clearly, this map is surjective. We now check that it is also injective:

$$\varphi(g_1Hg_1^{-1}) = \varphi(g_2Hg_2^{-1}) \implies g_1N(H) = g_2N(H) \implies g_1^{-1}g_2 \in N(H)$$
$$\implies g_1^{-1}g_2H(g_1^{-1}g_2)^{-1} = H \implies g_1^{-1}g_2Hg_2^{-1}g_1^{-1} = H \implies g_2Hg_2^{-1} = g_1Hg_1^{-1}$$

This shows that φ is a bijection, and the proof is complete.

Alternatively, we may define a map $\psi\colon T \to S$ by setting $\psi(gN(H)) = gHg^{-1}$. Before proceeding, we need to check this map is well defined, that is,

$$g_1N(H) = g_2N(H) \implies g_2Hg_2^{-1} = g_1Hg_1^{-1},$$

but this is done exactly as above. It is now clear that $\psi = \varphi^{-1}$; thus, both φ and ψ are bijections, and this completes the alternate proof.

3. Given a group G and an element $a \in G$, we define the *centralizer of* a to be the set $C(a)$ of elements $x \in G$ that commute with a; that is, $C(a) := \{g \in G : ga = ag\}$.

(i) Show that $C(a)$ is a subgroup of G.

Let $g, h \in C(a)$, so that $ga = ag$ and $ha = ah$. Then

$$(gh)a = g(ha) = g(ah) = (ga)h = (ag)h = a(gh) \implies gh \in C(a)$$
$$a = (g^{-1}g)a = g^{-1}(ga) = g^{-1}(ag) = (g^{-1}a)g \implies g^{-1}a = ag^{-1} \implies g^{-1} \in C(a),$$

thus showing that $C(a) \le G$.

(ii) Show that $\langle a\rangle \subseteq C(a)$.

Recall that $\langle a\rangle = \{a^k : k \in \mathbb{Z}\}$. Then $a^k \cdot a = a^{k+1} = a \cdot a^k$, and so $a^k \in C(a)$ for all $k \in \mathbb{Z}$, thus showing that $\langle a\rangle \subseteq C(a)$.

(iii) Show that $Z(G) \subseteq C(a)$.

Recall that $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$. In particular, if $g \in Z(G)$, then $ga = ag$, and so $Z(G) \subseteq C(a)$.

4. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$. Compute the factor groups $G/\langle(2,3)\rangle$ and $G/\langle(3,3)\rangle$. (In each case, write the result in terms of known finite groups, and explain your answer.)

Part 1. Note that $H = \langle(2,3)\rangle = \{(0,0),(2,3)\}$ is a group of order 2 (and thus isomorphic to $\mathbb{Z}_2$). Since G is an abelian group of order 24, the factor group G/H is an abelian group of order $\frac{|G|}{|H|} = \frac{24}{2} = 12$. Its elements are the cosets gH, where

$$g \in \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5),(1,0),(1,1),(1,2),(1,3),(1,4),(1,5)\}.$$

Note that G/H is generated by the element $(1,6)H$ of order 12; thus, $G/H \cong \mathbb{Z}_{12}$, the cyclic group of order 12.

Part 2. Note that $K = \langle (3,3) \rangle = \{(0,0),(3,3),(2,0),(1,3)\}$ is a cyclic group of order 4 (and thus isomorphic to $\mathbb{Z}_4$). Thus, the factor group G/K is an abelian group of order $\frac{|G|}{|K|} = \frac{24}{4} = 6$. Since there is only one abelian grooup of order 6, we must have $G/K \cong \mathbb{Z}_6$.

Alternatively, the elements of G/K are the cosets gK, where $g \in \{(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)\}$, and clearly this is the cyclic group $\langle (0,1)K \rangle$ of order 6, showing once again that $G/K \cong \mathbb{Z}_6$.

5. Let G be a group of order 35. Suppose G has precisely one subgroup of order 5 and one subgroup of order 7. Show that G is a cyclic group.

Let $a \in G$ be an element of order 5 and $b \in G$ an element of order 7. Then G contains the cyclic subgroups $\langle a \rangle = \{e, a, a^2, a^3, a^4\}$ and $\langle b \rangle = \{e, b, b^2, b^3, b^4, b^5, b^6\}$; thus, G contains the subset

$$S \coloneqq \langle a \rangle \cup \langle b \rangle = \{e, a, a^2, a^3, a^4, b, b^2, b^3, b^4, b^5, b^6\}$$

of size $11 = 5 + 7 - 1$. Since G has size $35 > 11$, there must be an element $g \in G$ such that $g \notin S$.

No, by Lagrange's Theorem, the order of g divides the size of G, and so $\operatorname{ord}(g) \in \{1,5,7,35\}$. Clearly, $\operatorname{ord}(g) \neq 1$, since otherwise $g = e \in S$. Also, $\operatorname{ord}(g) \neq 5$, since otherwise $\langle g \rangle$ would be a (cyclic) subgroup of G different from $\langle a \rangle$ (since $g \notin S$), thus contradicting our assumption that G has only one subgroup of order 5. The same reasoning shows that $\operatorname{ord}(g) \neq 7$.

The only remaining possibility is that $\operatorname{ord}(g) = 35$. Since $\langle g \rangle \subseteq G$ and $|G| = 35$, this implies $G = \langle g \rangle$, and so G is a cyclic group of order 35 (that is, $G \cong \mathbb{Z}_{35}$).