Midterm Exam

MATH 3175 – Group Theory

Due Thursday July 30

Problem 1. Describe all possible group homomorphisms $f\colon \mathbb{Z}_{15} \to D_6$ where D_6 is the symmetry group of a regular hexagon. Prove that your answer gives a complete list of all different such homomorphisms. Some results from previous homeworks might be of use.

Problem 2. Consider S_4, the symmetric group on $\{1, 2, 3, 4\}$. This group has 4 subgroups of order 6. Figure out what they are along with what group these subgroups are isomorphic to. Prove your answer. It may be wise to go back and review assignment 3.

Problem 3. The circle group is the set $T = \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with complex norm 1.

(1) Define a surjective homomorphism $\mathbb{R} \to T$, where $\mathbb{R}$ is considered as a group under addition. Hint: write the elements of T in polar coordinates and utilize Euler's formula.

(2) Show that $T \cong \mathbb{R}/\mathbb{Z}$.

Problem 4. Let G be the group of 3×3 upper-diagonal matrices with entries in $\mathbb{Z}_2$ and 1's down the diagonal:

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z}_2 \right\}.$$

(1) Show that G is a non-abelian group of order 8. What is the center of G?

(2) List all the subgroups of G, and indicate which ones are normal subgroups.

(3) Prove or disprove the following two assertions:

- G is isomorphic to Q_8, the quaternion group of order 8.
- G is isomorphic to D_4, the dihedral group of order 8.

Problem 5. For a group G and a subset $S \subset G$, define the *centralizer of S in G* (denoted $\mathrm{C}_G(S)$) and the *normalizer of S in G* (denoted $\mathrm{N}_G(S)$) as

$$\mathrm{C}_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\},$$
$$\mathrm{N}_G(S) = \{g \in G \mid gS = Sg\}.$$

(1) Show that both $\mathrm{C}_G(S)$ and $\mathrm{N}_G(S)$ are subgroups of G.

(2) Show that $\mathrm{C}_G(S)$ is a *normal* subgroup of $\mathrm{N}_G(S)$.

(3) Given an example showing that $\mathrm{C}_G(S)$ need not contain S.

(4) If $H \leq G$ is a subgroup of G, show that $\mathrm{N}_G(H)$ contains H.

Problem 6. Show that the following group isomorphisms hold:

$$\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong D_4\,,$$
$$\mathrm{Aut}(Q_8) \cong S_4\,.$$

That is, the automorphism group of the product of a cyclic group of order 2 with a cyclic group of order 4 is isomorphic to the dihedral group of order 8, while the automorphism group of the quaternion group of order 8 is isomorphic to the symmetric group of order 24. In each case, define a map between the two groups, and verify that this map is both a homomorphism and a bijection.

Problem 7. Here we will revisit commutators and Abelianizations in a more concrete way.

(1) Identity the commutator subgroups A_4' and S_4' and associate them to familiar isomorphism types. Furthermore, write out explicitly what their cosets are.

(2) Compute the respective Abelianizations for A_4 and S_4, and detail the behavior of the induced projection maps.