Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Summer 2, 2022**

Midterm Exam

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1. Prove the following statements.

(i) All cyclic groups are abelian.

(ii) All groups of prime order are cyclic.

(iii) Any two cyclic groups of the same size are isomorphic.

2. Let $G = \mathrm{GL}(2, 2)$ be the group of all invertible 2×2 matrices with entries in $\mathbb{Z}_2$, with group operation given my matrix multiplication.

(i) List all the elements of G and find their orders.

(ii) Does G contain a subgroup of order 3? Why, or why not?

(iii) Is G a cyclic group? Why, or why not?

(iv) Is G an abelian group? Why, or why not?

3. Consider the cyclic group $\mathbb{Z}_8 = \{[0]_8, \dots, [7]_8\}$ and the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. For each of these two groups:

(i) List all the subgroups, and display the information as a lattice of subgroups.

(ii) In each case, how many *distinct* subgroups are there?

(iii) In each case, how many *isomorphism classes* of subgroups are there?

(iv) In each case, how many *cyclic* subgroups are there?

4. Let G be a group, and let $H \leq G$ be a subgroup.

(i) Show that, for every element $a \in G$, the right coset Ha coincides (up to inversion in G) with the left coset $a^{-1}H$.

(ii) Use part (i) to construct a bijection between the set of right cosets of H and the set of left cosets of H.

(iii) Assume now that G is finite. Use part (ii) to show that the number of left cosets of H is equal to the number of right cosets of H.

[Note: We proved this in class as a corollary to Lagrange's theorem. This gives a different proof, but your argument should be independent of Lagrange's theorem!]

5. Let $\mathbb{C}^\times$ be the multiplicative group of non-zero complex numbers, and let $T = \{z \in \mathbb{C}^\times : |z| = 1\}$ be the subset of complex numbers with absolute value equal to 1.

(i) Show that T is a subgroup of $\mathbb{C}^\times$.

(ii) Sketch T in the x-y plane (where recall $z = x + iy \in \mathbb{C}$ corresponds to the point in $\mathbb{R}^2$ with coordinates (x, y).)

(iii) Describe the (right) cosets of T in geometric terms and sketch at least 4 of these cosets, labelling each one accordingly.

6. Let G be a group of order 21. Suppose that G has precisely one subgroup of order 3, and one subgroup of order 7. Show that G is cyclic.

7. Let $\varphi\colon G \to H$ be a homomorphism. Prove the following:

(i) If φ is injective, then $|G|$ divides $|H|$.

(ii) If φ is surjective, and G is abelian, then H is also abelian.

(iii) If φ is surjective, and G is cyclic, then H is also cyclic.

8. For each of the following pairs of groups, decide whether they are isomorphic or not. In each case, give a brief reason why.

(i) $\mathbb{Z}_9^\times$ and $\mathbb{Z}_8$.

(ii) $\mathbb{Z}_{16}^\times$ and $\mathbb{Z}_8$.

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_6$.

(iv) $\mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$.