Prof. Alexandru Suciu

MATH 3175 | **Group Theory** | **Summer 2, 2022**

Assignment 3

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1. Let $G = \langle a \rangle$ be a finite cyclic group of order n.

(i) For an element $a^k \in G$ with $0 < k < n$, show that the order of a^k is equal to the order of the cyclic subgroup $\langle a^k \rangle$.

(ii) Show that $\langle a^k \rangle = \{a^{ks} : s \in \mathbb{Z}\} = \{a^{ks}a^{nt} : s, t \in \mathbb{Z}\}$.

(iii) Let $d = \gcd(n, k)$. Use parts (i) and (ii) to show that

$$\operatorname{ord} a^k = n/d.$$

2. Let G be a cyclic group of size at least 3.

(i) Show that G has at least 2 distinct generators.

(ii) If G is finite, show that G has an even number of distinct generators.

3. For each of the following groups, find all their cyclic subgroups:

(i) $\mathbb{Z}_{14}^{\times}$.

(ii) $\mathbb{Z}_{20}^{\times}$.

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_6$.

4. Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8. Find all the subgroups of Q_8 and draw the corresponding lattice of subgroups.

5. Let $H = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$.

(i) Sketch H in the plane.

(ii) Consider $\mathbb{R}^2$ as a group under vector addition. Show that H is a subgroup of $\mathbb{R}^2$. Is H commutative?

(iii) Describe the cosets of H in geometric terms and make a sketch of a few of the cosets.

6. Let S_4 be the group of permutations of the set $\{1, 2, 3, 4\}$. Consider the subgroup H generated by the cyclic permutation $(1\ 2\ 3\ 4) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{smallmatrix}\right)$.

(i) Write down all the right cosets and all the left cosets of H in S_4. (Make sure to indicate all the elements in each coset.)

(ii) What is the index of H in S_4?