Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Summer 2, 2022**

Assignment 2

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1. Let R be a ring. An element $x \in R$ is called an *idempotent* if $x^2 = x$. (For instance, both 0 and 1 are idempotents.)

(i) Let x be an idempotent, $x \neq 1$. Show that x is a zero-divisor.

(ii) The ring R is called a *Boolean ring* if every element in R is an idempotent. Show that in such a ring, the following identities hold:

$$x = -x \qquad \text{for all } x \in R, \tag{1}$$

$$xy = yx \qquad \text{for all } x, y \in R. \tag{2}$$

2. For the ring $R = \mathbb{Z}_{12}$:

(i) List all the invertible elements, zero-divisors, and idempotents.

(ii) Are there any elements which are neither zero-divisors nor invertible?

(iii) Are there any zero-divisors which are not idempotent?

3. Let $(G, \cdot, e)$ be a group. An element $a \in G$ is said to have finite order if there is a positive integer n such that $a^n := a \cdot a \cdots a$ (multiplication done n times) is equal to the identity e. The smallest such n is called the *order* of a, and is denoted by $\operatorname{ord}(a)$ (or $o(a)$, or $|a|$). If no such n exists, we say a has infinite order, and write $\operatorname{ord}(a) = \infty$.

(i) Show that, for all $a, b \in G$,

(1) $\operatorname{ord}(a) = \operatorname{ord}(a^{-1})$.

(2) $\operatorname{ord}(ab) = \operatorname{ord}(ba)$.

(ii) Assume now that the orders of a and b are finite and coprime, and that $ab = ba$. Show that $\operatorname{ord}(ab) = \operatorname{ord}(a)\operatorname{ord}(b)$.

4. For each of the following groups, list all their elements, together with their orders:

(i) $\mathbb{Z}_{12}$.

(ii) $\mathbb{Z}_{12}^{\times}$.

(iii) $\mathbb{Z}_6 \times \mathbb{Z}_2$.

(iv) $S_3 \times \mathbb{Z}_2$.

5. Let G be the set of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, with $a, b \in \mathbb{R}$ and $a \neq 0$.

(i) Show that G is a group under matrix multiplication.

(ii) Is G abelian?

(iii) Find all the elements of G that commute with $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.