Prof. Alexandru Suciu

MATH 3175 | **Group Theory** | **Summer 2, 2022**

Assignment 1

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1. Consider the binary operations $*$ and $\star$ on the set $S = \{e, a, b, c, d\}$ given by the following multiplication tables:

$*$	e	a	b	c	d
e	e	a	b	c	d
a	a	b	d	e	c
b	b	d	c	a	e
c	c	e	a	d	b
d	d	c	e	b	a

$\star$	e	a	b	c	d
e	e	a	b	c	d
a	a	c	e	d	b
b	b	d	c	a	e
c	c	e	d	b	a
d	d	b	a	e	c

Which (if either) of these binary operations gives S the structure of a group? Prove your answer.

2. Let G a group.

(i) Suppose $(ab)^{-1} = a^{-1}b^{-1}$, for all a and b in G. Prove that G is abelian.

(ii) Give an example of a group G and two elements $a, b \in G$ for which $(ab)^{-1} \neq a^{-1}b^{-1}$.

3. Let G and H be two groups, and let $G \times H$ be their product.

(i) If both G and H are commutative, show that $G \times H$ is also commutative.

(ii) If either G or H is non-commutative, show that $G \times H$ is non-commutative.

4. Let G be a group, with group operation $\cdot$ and identity $e = 1$. Let u be an element not in G and consider the magma

$$M = G \cup (Gu),$$

where $Gu = \{gu \mid g \in G\}$ and the product in M is given by the usual product of elements in G, together with $1 \cdot u = u$ and

$$(gu)h = (gh^{-1})u$$
$$g(hu) = (hg)u$$
$$(gu)(hu) = h^{-1}g.$$

(i) Show that $u^2 = 1$ and $ug = g^{-1}u$.

(ii) Show that M has an identity.

(iii) Show that the multiplication on M is associative if and only if G is abelian.

5. Consider the set of matrices $S = \{I, A, B, C\}$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Write out the multiplication table for S.

(ii) Show that the set S (with this multiplication) is a magma. Is this magma abelian?

(iii) Is the magma S a group?

6. Give an example of three permutations $\alpha, \beta, \gamma \in S_4$ (none of which is equal to the identity permutation) such that $\alpha\beta = \beta\alpha$ and $\beta\gamma = \gamma\beta$ but $\alpha\gamma \neq \gamma\alpha$.