Prof. Alexandru Suciu

MATH 3175 | **Group Theory** | **Summer 2, 2022**

Final Exam

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1. Consider the following functions.

(a) $f\colon \mathbb{R}^\times \to \mathrm{GL}_2(\mathbb{R})$, $f(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$.

(b) $f\colon \mathbb{R}^\times \to \mathrm{GL}_2(\mathbb{R})$, $f(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$.

(c) $f\colon\ \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}$, $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ab$.

(d) $f\colon\ \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}$, $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$.

(e) $f\colon\ \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$, $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$.

For each of these functions, answer the following questions, with a (brief) justification.

(i) Is f a homomorphism?

(ii) Is f injective?

(iii) Is f surjective?

2. Let S_4 be the group of all permutations of the set $\{1, 2, 3, 4\}$. Consider the subgroups S_3 of all permutations fixing 4.

(i) Write down all the *left* and *right* cosets of S_3 in S_4. Be sure to indicate the elements of each coset.

(ii) What is the index of S_3 in S_4?

(iii) Is S_3 a normal subgroup of S_4? Why or why not?

3. Let D_n ($n \geq 3$) be the dihedral group of order $2n$.

(i) Show that $D_{10} \cong D_5 \times \mathbb{Z}_2$ by constructing an explicit isomorphism between the two groups.

(ii) What are the centers of D_5 and D_{10}?

(iii) Identify the quotient groups $D_5/Z(D_5)$ and $D_{10}/Z(D_{10})$ in terms of known groups.

4. Let GL$(2, 11)$ be the group of all invertible 2×2 matrices with entries in $\mathbb{Z}_{11}$, with group operation given my matrix multiplication. Consider the following two matrices in this group (where an entry listed as k is shorthand for $[k]_{11}$):

$$A = \begin{pmatrix} 3 & 10 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 10 \\ 8 & 8 \end{pmatrix}.$$

(i) Show that A has order 5, B has order 2, and that $BAB^{-1} = A^{-1}$.

(ii) Consider the subset of GL$(2, 11)$ given by

$$G = \{A^m B^n : m, n \in \mathbb{Z}\}.$$

Show that G is a subgroup of GL$(2, 11)$.

(iii) List all the elements of G, together with their orders.

(iv) Identify G in terms of known groups.

5. Let $G = \mathbb{Z}_8 \times \mathbb{Z}_6$, and consider the subgroups $H = \{(0, 0), (4, 0), (0, 3), (4, 3)\}$ and $K = \langle (2, 2) \rangle$. Identify the following groups (as direct products of cyclic groups of prime power order):

(i) H and G/H.

(ii) K and G/K.

6. Recall that, for every group G, the map $\varphi \colon G \to \mathrm{Sym}(G)$ which sends $g \in G$ to the bijection $\ell_g \colon G \to G$, $\ell_g(x) = gx$ is an injective homomorphism. Consider now the quaternion group $G = Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$. Identifying $\mathrm{Sym}(Q_8)$ with S_8 leads to an embedding, $\varphi \colon Q_8 \hookrightarrow S_8$.

(i) List the 8 permutations $\varphi(g)$, where g runs through the elements of Q_8.

(ii) Are -1 and i conjugate in Q_8? If yes, find an element $g \in Q_8$ that conjugates one to the other; if not, explain why not.

(iii) Are $\varphi(-1)$ and $\varphi(i)$ conjugate in S_8? If yes, find an element $\tau \in S_8$ that conjugates one to the other; if not, explain why not.

(iv) Are i and j conjugate in Q_8? If yes, find an element $g \in Q_8$ that conjugates one to the other; if not, explain why not.

(v) Are $\varphi(i)$ and $\varphi(j)$ conjugate in S_8? If yes, find an element $\tau \in S_8$ that conjugates one to the other; if not, explain why not.

7. Let G be a group of odd order, and let N be a normal subgroup of order 5. Show that N is contained in the center of G.

8. Let G be a non-abelian group of order p^3, where p is a prime. Show that the center of G has order p.