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MATH 3175 **Group Theory** **Fall 2010**

Quiz 6

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1. Let H be set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$, with $a, c, d \in \mathbb{Z}$ and $ad = \pm 1$.

(a) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{Z})$.

H is a subset of $\mathrm{GL}_2(\mathbb{Z})$ and

- The identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$.
- Closed under multiplication: $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \begin{bmatrix} a' & 0 \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' & 0 \\ ca' + dc' & dd' \end{bmatrix} \in H$.
- Closed under taking inverse: $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} = (ad)^{-1} \begin{bmatrix} d & 0 \\ -c & a \end{bmatrix} \in H$.

Hence, H is a subgroup.

(b) Is H a normal subgroup of $\mathrm{GL}_2(\mathbb{Z})$?

No. Because, for instance, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \notin H$.

(**NOTE**: $\mathrm{GL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}$)

2. Let $G = U(16)$, and $H = \{1, 15\}$.

(a) List the elements of G/H.

$G/H = \{H, 3H, 5H, 7H\}$

(b) Compute the Cayley table for this group.

	H	$3H$	$5H$	$7H$
H	H	$3H$	$5H$	$7H$
$3H$	$3H$	$7H$	H	$5H$
$5H$	$5H$	H	$7H$	$3H$
$7H$	$7H$	$5H$	$3H$	H

3. Let $G = \mathbb{Z}_4 \oplus \mathbb{Z}_2$, and consider the subgroup $H = \{(0,0), (2,0), (0,1), (2,1)\}$.

(a) Identify the group H.

H is an abelian group of order 4, hence it is isomorphic to either $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4$. There are no elements in H with order 4, so it cannot be isomorphic to $\mathbb{Z}_4$. So, $H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(b) Show that H is a normal subgroup of G.

The group G is abelian, thus all of its subgroups are normal. Hence, H is a normal subgroup.

(c) Identify the group G/H.

$$|G/H| = \frac{|G|}{|H|} = 8/4 = 2.$$

So, $G/H \cong \mathbb{Z}_2$.

4. Let $\mathbb{R}^*$ be the multiplicative group of non-zero real numbers, and let $\phi\colon \mathbb{R}^* \to \mathbb{R}^*$ be the function given by $\phi(x) = x^2$.

(a) Show that ϕ is a homomorphism.

$\phi(xy) = (xy)^2 = x^2y^2 = \phi(x)\phi(y)$.

(b) Find $\ker(\phi)$ and $\operatorname{im}(\phi)$.

$\ker(\phi) = \{x \in \mathbb{R}^* | \phi(x) = 1\} = \{\pm 1\}$,
$\operatorname{im}(\phi) = \{y \in \mathbb{R}^* \mid \exists x \text{ such that } \phi(x) = y\} = \{y \in \mathbb{R} \mid y > 0\}$.

5. Suppose $\phi\colon \mathbb{Z}_{20} \to \mathbb{Z}_{12}$ is a homomorphism with $\phi(3) = 9$.

(a) Determine $\phi(x)$, for all $x \in \mathbb{Z}_{20}$.

$\phi(1) = \phi(21) = \phi(3 \times 7) = \phi(3+3+3+3+3+3+3) = 7\phi(3) = 7 \times 9 = 3$, so $\phi(x) = 3x$.

(b) Find $\ker(\phi)$ and $\operatorname{im}(\phi)$.

$\ker(\phi) = \{x \in \mathbb{Z}_{20} \mid \phi(x) = 0\} = \{0, 4, 8, 12, 16\}$
$\operatorname{im}(\phi) = \{0, 3, 6, 9\}$

6. Show that there is no surjective homomorphism from $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$ onto $\mathbb{Z}_9 \oplus \mathbb{Z}_9$.

Suppose there is a homomorphism ϕ from $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$ onto $\mathbb{Z}_9 \oplus \mathbb{Z}_9$. Since the two groups have the same order (namely, 81), the kernel is trivial, which means $\ker\phi = \{(0, 0)\}$. So ϕ is injective. Hence, ϕ is a bijection and a homomorphism, that is, ϕ is an isomorphism.

But $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$ has an element of order 27, while the other group does not. Thus, the two groups cannot be isomorphic.

Our assumption has led to a contradiction. Thus, there cannot be any surjective homomorphism $\phi\colon \mathbb{Z}_{27} \oplus \mathbb{Z}_3 \to \mathbb{Z}_9 \oplus \mathbb{Z}_9$.