Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Fall 2010**

Solutions to Quiz 2

---

1. Let G be the group defined by the following Cayley table.

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	4	1	6	7	8	5
3	3	4	1	2	7	8	5	6
4	4	1	2	3	8	5	6	7
5	5	8	7	6	1	4	3	2
6	6	5	8	7	2	1	4	3
7	7	6	5	8	3	2	1	4
8	8	7	6	5	4	3	2	1

(a) For each element $a \in G$, find the order $|a|$.

k	1	2	3	4	5	6	7	8
$\|k\|$	1	4	2	4	2	2	2	2

(b) What is the center of G?

$Z(G) = \{1, 3\}$

2. Let G be an abelian group with identity e, and let H be the set of all elements $x \in G$ that satisfy the equation $x^3 = e$. Prove that H is a subgroup of G.

Pf.

- $e^3 = e$, hence $e \in H$.
- If $a, b \in H$, then $(ab)^3 = ababab = a^3b^3 = ee = e$. The second equality holds because the group G is abelian. So $ab \in H$.
- If $a \in H$, that is $a^3 = e$, multiply both sides of the equlity by $(a^{-1})^3$, we will get $e = (a^{-1})^3$. Hence $a^{-1} \in H$.

in conclusion, H is a subgroup of G.

3. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, viewed as a 2×2 matrix with entries in $\mathbb{Z}_5$.

(a) Show that A belongs to $\mathrm{GL}_2(\mathbb{Z}_5)$.

$\det A = 2 \times 2 - 1 \times 1 = 3 \neq 0$. Hence $A \in \mathrm{GL}_2(\mathbb{Z}_5)$.

(b) Does A belong to $\mathrm{SL}_2(\mathbb{Z}_5)$? Why, or why not?

$A \notin \mathrm{SL}_2(\mathbb{Z}_5)$ because $\det A \neq 1$.

(c) Find all the elements in the cyclic subgroup $\langle A \rangle$ generated by A.

$$A, A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}, A^4 = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, $\langle A \rangle = \{I, A, A^2, A^3\}$.

(d) Find the order of A in $\mathrm{GL}_2(\mathbb{Z}_5)$.

Since there are four elements in $\langle A \rangle$, so the order of A is 4.

4. Let G be a group, H a subgroup of G, and a an element of H. Recall $C(a)$ denotes the centralizer of a, whereas $C(H)$ denotes the centralizer of H.

(a) Show that $C(H) \subseteq C(a)$.

Pf. For any $x \in C(H)$, $xh = hx \ \forall h \in H$.
Since $a \in H$, $xa = ax$. Hence $x \in C(a)$.
So $C(H) \subseteq C(a)$.

(b) Suppose $H = \langle a \rangle$ is the cyclic subgroup generated by a. Show that $C(\langle a \rangle) = C(a)$.

Pf.

- $a \in \langle a \rangle$, so by (a), $C(\langle a \rangle) \subseteq C(a)$.
- For any $x \in C(a)$, $xa = ax$, also $a^{-1}x = xa^{-1}$.
  So for any $k \in \mathbb{Z}$, $xa^k = a^k x$. Hence $x \in C(\langle a \rangle)$.
  So $C(a) \subseteq C(\langle a \rangle)$

In conclusion, $C(\langle a \rangle) = C(a)$.

5. Consider the group $G = \mathbb{Z}_{18}$, with group operation addition modulo 18.

(a) For each element $k \in \mathbb{Z}_{18}$, compute the order of k.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\|k\|$	1	18	9	6	9	18	3	18	9	2	9	18	3	18	9	6	9	18

(b) Find all the generators of $\mathbb{Z}_{18}$.

One element is a generators of G if and only if its order is 18.
(Alternative interpretation: One element n is a generators of $G = \mathbb{Z}_1 8$ if and only if $\gcd(n, 18) = 1$.)
So the generators are 1, 5, 7, 11, 13 and 17.

(c) Write all the elements of the subgroup $\langle 3 \rangle$.

$\langle 3 \rangle = \{0, 3, 6, 9, 12, 15\}$

(d) Find all the generators of $\langle 3 \rangle$.

Since the number of elements in $\langle 3 \rangle$ is 6, one is a generator of $\langle 3 \rangle$ if and only if its order is 6. So it has two generators 3 and 15

6. Let $G = \langle a \rangle$ be a group generated by an element a of order $|a| = 28$.

(a) Is $\langle a \rangle = \langle a^{-1} \rangle$? Is a^{-1} a generator of G? Justify your answers.

- $\langle a^k \rangle \subseteq \langle a \rangle$ for all $k \in \mathbb{Z}$, so $\langle a^{-1} \rangle \subseteq \langle a \rangle$.
- $a = (a^{-1})^{-1}$, so $\langle a \rangle \subseteq \langle a^{-1} \rangle$.

So $\langle a^{-1} \rangle = \langle a \rangle$.
And a^{-1} is a generator of G.

(b) Find all elements of G which generate G.

a^k is a generator of G if and only if $\gcd(k, 28) = 1$.
So the generators are a, a^3, a^5, a^9, a^{11}, a^{13}, a^{15}, a^{17}, a^{19}, a^{23}, a^{25} and a^{27}.

(c) Find an element in G that has order 4. Does this element generate G?

a^7 has order 4. Since its order is not 28, so it doesn't generate G.

(d) Find the order of a^{12}.

$$|a^{12}| = \frac{28}{\gcd(28, 12)} = \frac{28}{4} = 7.$$