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MATH 3175 **Group Theory** **Fall 2010**

Answers to Problems on Practice Quiz 5

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1. Prove, by comparing orders of elements, that the following pairs of groups are not isomorphic:

(a) $\mathbb{Z}_8 \oplus \mathbb{Z}_4$ and $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$.

There is an element of order 16 in $\mathbb{Z}_{16} \oplus \mathbb{Z}_2$, for instance, $(1, 0)$, but no element of order 16 in $\mathbb{Z}_8 \oplus \mathbb{Z}_4$.

(b) $\mathbb{Z}_9 \oplus \mathbb{Z}_9$ and $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$.

There is an element of order 27 in $\mathbb{Z}_{27} \oplus \mathbb{Z}_3$, for instance, $(1, 0)$, but no element of order 27 in $\mathbb{Z}_9 \oplus \mathbb{Z}_9$.

2. Describe a specific isomorphism $\phi\colon \mathbb{Z}_6 \oplus \mathbb{Z}_5 \to \mathbb{Z}_{30}$.

Set $\phi((1, 1)) = 1$, and then use the fact that ϕ is a homomorphism to determine $\phi((i, j))$.

3. Describe a specific isomorphism $\psi\colon U(16) \to \mathbb{Z}_2 \oplus \mathbb{Z}_4$.

$$\begin{aligned} 1 &\mapsto (0,0) \\ 3 &\mapsto (0,1) \\ 5 &\mapsto (1,1) \\ 7 &\mapsto (1,0) \\ 9 &\mapsto (0,2) \\ 11 &\mapsto (0,3) \\ 13 &\mapsto (1,3) \\ 15 &\mapsto (1,2) \end{aligned}$$

4. Prove or disprove that $D_6 \cong D_3 \oplus \mathbb{Z}_2$.

Yes, the two groups are isomorphic. Why?

5. Prove or disprove that $D_{12} \cong D_4 \oplus \mathbb{Z}_3$.

Hint: count elements of order 2

6. How many elements of order 6 are there in $\mathbb{Z}_6 \oplus \mathbb{Z}_9$?

The order of (a, b) is the least common multiple of the order of a and that of b. We would like the order of (a, b) to be 6. This can happen only if the order of a is 6 and that of b is 1 or 3, or the order of a is 2 and that of b is 3. The desired elements of order 6 are:

$$(1, 0), (5, 0), (1, 3), (1, 6), (5, 3), (5, 6), (3, 3), (3, 6)$$

7. How many elements of order 25 are there in $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$?

The number of elements of order 25 in $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ equals

$$1 \times \phi(25) + \phi(5) \times \phi(25) = (25 - 5) + (5 - 1) \times (25 - 5) = 100.$$

Note 1: The number of elements of order 5 equals $\phi(25) + \phi(5) = (25-5) + (5-1) = 24$. Accounting also for the single element of order 1, namely the identity $(0, 0)$, we have in all $100 + 24 + 1 = 125$ elements $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$, as we should (check: $5 \cdot 25 = 125$).

Note 2: We used here the fact that $\phi(p^n) = p^n - p^{n-1}$ for any odd prime p, which follows from the corresponding fact about $U(p^n)$ mentioned in the solution to Problem 13 below.

8. How many elements of order 3 are there in $\mathbb{Z}_{300000} \oplus \mathbb{Z}_{900000}$?

$1 \times \phi(3) + \phi(3) \times \phi(3) + \phi(3) \times 1 = 8$

9. Let p be a prime. Determine the number of elements of order p in $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$.

$1 \times \phi(p) + \phi(p) \times \phi(p) + \phi(p) \times 1 = p^2 - 1$

10. Let $G = S_3 \oplus \mathbb{Z}_5$. What are all possible orders of elements in G? Prove that G is *not* cyclic.

Possible orders: $1, 2, 3, 5, 10, 15$

The order of G is 30. There is no element of order 30 in the group, so G is not cyclic.

11. The group $S_3 \oplus \mathbb{Z}_2$ is isomorphic to one of the following groups: $\mathbb{Z}_{12}$, $\mathbb{Z}_6 \oplus \mathbb{Z}_2$, A_4, D_6. Determine which one, by a process of elimination.

The group $S_3 \oplus \mathbb{Z}_2$ is not abelian, but $\mathbb{Z}_{12}$ and $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ are.
The elements of $S_3 \oplus \mathbb{Z}_2$ have order 1, 2, 3, or 6, whereas the elements of A_4 have order 1, 2, or 3.
So what's the conclusion?

12. Describe all abelian groups of order $1{,}008 = 2^4 \cdot 3^2 \cdot 7$. Write each such group as a direct product of cyclic groups of prime power order.

$\mathbb{Z}_{2^4} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_{2^4} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_2 \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^3} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_{2^2} \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$,
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_7$, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$

13. Describe $U(1{,}008)$ as a direct product of cyclic groups.

Some general facts worth knowing:

$$
\begin{aligned}
U(m\cdot n) &\cong U(m)\oplus U(n) && \text{if } \gcd(m,n)=1\\
U(2)\cong\{0\}, &\qquad U(4)\cong\mathbb{Z}_2 &&\\
U(2^n) &\cong \mathbb{Z}_2\oplus\mathbb{Z}_{2^{n-2}} && \text{for all } n\geq 3\\
U(p^n) &\cong \mathbb{Z}_{p^n-p^{n-1}} && \text{for any odd prime } p
\end{aligned}
$$

Hence:

$$
\begin{aligned}
U(1008) &\cong U(2^4)\oplus U(3^2)\oplus U(7)\\
&\cong (\mathbb{Z}_2\oplus\mathbb{Z}_4)\oplus\mathbb{Z}_6\oplus\mathbb{Z}_6\\
&\cong \mathbb{Z}_2^3\oplus\mathbb{Z}_4\oplus\mathbb{Z}_3^2
\end{aligned}
$$

14. Describe $U(195)$ as a direct product of cyclic groups in four different ways.

$$
\begin{aligned}
U(195) &\cong U(3)\oplus U(5)\oplus U(13)\\
&\cong \mathbb{Z}_2\oplus\mathbb{Z}_4\oplus\mathbb{Z}_{12}\\
&\cong \mathbb{Z}_2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_4\oplus\mathbb{Z}_4\\
&\cong \mathbb{Z}_6\oplus\mathbb{Z}_4\oplus\mathbb{Z}_4
\end{aligned}
$$

15. For each of the following groups, compute the number of elements of order 1, 2, 4, 8, and 16:

$$\mathbb{Z}_{16},\quad \mathbb{Z}_8\oplus\mathbb{Z}_2,\quad \mathbb{Z}_4\oplus\mathbb{Z}_4,\quad \mathbb{Z}_4\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2.$$

Group\Order	1	2	4	8	16
$\mathbb{Z}_{16}$	1	1	2	4	8
$\mathbb{Z}_8\oplus\mathbb{Z}_2$	1	3	4	8	0
$\mathbb{Z}_4\oplus\mathbb{Z}_4$	1	3	12	0	0
$\mathbb{Z}_4\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$	1	7	8	0	0

16. List all abelian groups (up to isomorphism) of order $160 = 2^5\cdot 5$.

$\mathbb{Z}_{2^5}\oplus\mathbb{Z}_5$
$\mathbb{Z}_2\oplus\mathbb{Z}_{2^4}\oplus\mathbb{Z}_5$
$\mathbb{Z}_{2^2}\oplus\mathbb{Z}_{2^3}\oplus\mathbb{Z}_5$
$\mathbb{Z}_2\oplus\mathbb{Z}_{2^2}\oplus\mathbb{Z}_{2^2}\oplus\mathbb{Z}_5$
$\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2^3}\oplus\mathbb{Z}_5$
$\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_{2^2}\oplus\mathbb{Z}_5$
$\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_5$

16'. List all abelian groups (up to isomorphism) of order $360 = 2^3 \cdot 3^2 \cdot 5$.

$\mathbb{Z}_{2^3} \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{360}$
$\mathbb{Z}_{2^2} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{180} \oplus \mathbb{Z}_2$
$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{3^2} \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{90} \oplus \mathbb{Z}_2^2$
$\mathbb{Z}_{2^3} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_3$
etc

17. (a) List the five partitions of 4, and the abelian groups of order 81 that correspond to them.

$4 = 1 + 3 = 2 + 2 = 1 + 1 + 2 = 1 + 1 + 1 + 1$

$\mathbb{Z}_{81}, \quad \mathbb{Z}_3 \oplus \mathbb{Z}_{27}, \quad \mathbb{Z}_9 \oplus \mathbb{Z}_9, \quad \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9, \quad \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$

(b) A certain abelian group G of order 81 has no elements of order 27, and 54 elements of order 9. Which group is it? Why?

$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_9$

18. How many abelian groups (up to isomorphism) are there

(a) of order 21? One: $\mathbb{Z}_{21}$

(b) of order 105? One: $\mathbb{Z}_{105}$

(c) of order 210? One: $\mathbb{Z}_{210}$

(d) of order 25? $25 = 5 \times 5$, so there are two, $\mathbb{Z}_{25}$ and $\mathbb{Z}_5 \oplus \mathbb{Z}_5$

(e) of order 125? Use: $125 = 5 \times 25 = 5 \times 5 \times 5$

(f) of order 625? Use: $625 = 5 \times 125 = 25 \times 25 = 5 \times 5 \times 25 = 5 \times 5 \times 5 \times 5$

19. Let G be a finite abelian group of order n.

(a) Suppose n is divisible by 10. Show that G has a cyclic subgroup of order 10.

According to the decomposition theorem for finite abelian groups, G contains the group $\mathbb{Z}_2 \oplus \mathbb{Z}_5$ as a subgroup, which is cyclic of order 10.

(b) Suppose n is divisible by 9. Show, by example, that G need not have a cyclic subgroup of order 9.

Take $G = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.

20. Suppose G is an abelian group of order 168, and that G has exactly three elements of order 2. Determine the isomorphism class of G.

$G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$.