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MATH 3175 **Group Theory** **Fall 2010**

Solutions to Practice Quiz 4

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1. Write down all the automorphisms of the group $\mathbb{Z}_5$.

The automorphisms are $\phi_k\colon \mathbb{Z}_5 \to \mathbb{Z}_5$, with $\phi_k(x) = kx$, for $k = 1, 2, 3, 4$.

2. Let $\mathbb{R}^+$ be the multiplicative group of positive real numbers. Show that the map $x \mapsto \sqrt[3]{x}$ is an automorphism of $\mathbb{R}^+$.

Let $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be the function given by $\phi(x) = \sqrt[3]{x}$. We need to show that ϕ is a bijection, and a homomorphism.

- ϕ injective. Suppose $\sqrt[3]{x} = \sqrt[3]{y}$. Taking cubes on both sides, we get $x = y$.
- ϕ surjective. Let $y \in \mathbb{R}^+$. Then $y = \phi(x)$, where $x = y^3$.
- ϕ a homomorphism. Compute: $\phi(x \cdot y) = \sqrt[3]{x \cdot y} = \sqrt[3]{x} \cdot \sqrt[3]{y} = \phi(x) \cdot \phi(y)$.

3. Show that the map $x \mapsto e^x$ is an isomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \cdot)$.

Let $\phi\colon \mathbb{R} \to \mathbb{R}^+$ be the function given by $\phi(x) = e^x$. We need to show that ϕ is a bijection, and a homomorphism.

- ϕ injective. Suppose $e^x = e^y$. Taking natural logs on both sides, we get $x = y$.
- ϕ surjective. Let $y \in \mathbb{R}^+$. Then $y = \phi(x)$, where $x = \log y$.
- ϕ a homomorphism. Compute: $\phi(x + y) = e^{x+y} = e^x \cdot e^y = \phi(x) \cdot \phi(y)$.

4. For each the following pair of groups, decide whether they are isomorphic or not. In each case, give a brief reason why.

(a) $U(5)$ and $U(10)$.
Yes. They are both cyclic of order 4.

(b) $U(8)$ and $\mathbb{Z}_4$.
No. $U(8)$ doesn't has an element of order 4, but $\mathbb{Z}_4$ does.

(c) $U(10)$ and $\mathbb{Z}_4$.
Yes. They are both cyclic of order 4.

(d) S_3 and $\mathbb{Z}_6$.
No. S_3 is not abelian, but $\mathbb{Z}_6$ is.

(e) S_3 and D_3.
Yes. They are both the permutation group of the three vertices of a triangle.

(f) A_4 and D_6.
No. A_4 doesn't has an element of order 6, but D_6 does.

5. Let $\phi\colon G \to H$ be an isomorphism between two groups. Suppose G is abelian. Show that H is also abelian.

Let $x, y \in H$. Since ϕ is a surjection, there are elements $a, b \in G$ such that $x = \phi(a)$ and $y = \phi(b)$. Since ϕ is a homomorphism, and G is abelian, we have:

$$xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx.$$

Hence, H is abelian.

6. Let g and h be two elements in a group G, and let ϕ_g and ϕ_h be the corresponding inner automorphisms. Suppose $\phi_g = \phi_h$. Show that $h^{-1}g$ belongs to the center of G.

By definition, $\phi_g(a) = gag^{-1}$ and $\phi_h(a) = hah^{-1}$, for any $a \in G$. Thus, the condition $\phi_g = \phi_h$ means $gag^{-1} = hah^{-1}$, for all $a \in G$. Hence,

$$(h^{-1}g)a(h^{-1}g)^{-1} = h^{-1}gag^{-1}h = h^{-1}hah^{-1}h = a.$$

Since this holds for any $a \in G$, we conclude that $h^{-1}g$ is in the center of G.

7. Let G be a finite group, H a subgroup of G, and K a subgroup of H. Show that $|G : K| = |G : H| \cdot |H : K|$.

Using Lagrange's Theorem, we get:

$$|G : K| = \frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = |G : H| \cdot |H : K|\,.$$

8. Let G be a group, and let a be an element of order 24. How many left cosets of $\langle a^{10}\rangle$ in $\langle a\rangle$ are there? List all these cosets.

The order of a^{10} is $24/\gcd(10, 24) = 24/2 = 12$. So the number of left cosets is $|\langle a\rangle| \,/\, |\langle a^{10}\rangle| = 24/12 = 2$. These cosets are:

$$\langle a^{10}\rangle = \{e, a^{10}, a^{20}, a^{16}, a^{2}, \dots\},$$
$$a \cdot \langle a^{10}\rangle = \{a, a^{11}, a^{21}, a^{17}, a^{3}, \dots\}.$$

9. Let D_4 be dihedral group of order 8 (the group of symmetries of the square), let $H = \langle R_1\rangle$ be the subgroup generated by a counter-clockwise rotation by 90°, and let $K = \langle S_0\rangle$ be the subgroup generated by a reflection across the horizontal axis.

(a) Write down all the left cosets of H in D_4.
H, S_0H

(b) Write down all the right cosets of H in D_4.
H, HS_0

(c) Write down all the left cosets of K in D_4.
K, R_1K, R_1^2K, R_1^3K

(d) Write down all the right cosets of K in D_4.
K, KR_1, KR_1^2, KR_1^3

(e) Compute the indices $|D_4 : H|$ and $|D_4 : K|$.
2, 4

10. Let S_4 be the group of permutations of the set $\{1, 2, 3, 4\}$, and let A_4 the subgroup of even permutations.

(a) Write down all the left cosets of A_4 in S_4.
A_4, $(1, 2)A_4$

(b) Write down all the right cosets of A_4 in S_4.
A_4, $A_4(1, 2)$

(c) What is the index of A_4 in S_4?
$|S_4 : A_4| = |S_4| \,/\, |A_4| = 4!/(4!/2) = 2.$

11. Suppose a group contains elements of orders 1 through 9. What is the minimum possible order of the group?

The order of the group is divisible by $8, 9, 5$ and 7. So it is at least $\mathrm{lcm}(8, 9, 5, 7) = 8 \times 9 \times 5 \times 7 = 2,520$. And $\mathbb{Z}_{2,520}$ satisfies the stated condition. So $2,520$ is the minimum possible order for such a group.

12. Suppose K is a subgroup of H, and H is a subgroup of G. If $|K| = 30$ and $|G| = 300$, what are the possible values for $|H|$?

By Lagrange's Theorem, we have: $|K| = 30$ divides $|H|$ and $|H|$ divides $|G| = 300$. So the possible values of $|H|$ are 30, 60,150, and 300.

13. Suppose $|G| = 21$, and G has precisely one subgroup of order 3, and one subgroup of order 7. Show that G is cyclic.

Suppose the subgroup of order 3 is H, and the one of order 7 is K, then there are $1 + 2 + 6 = 9$ elements in $H \cup K$. Choose an element a from $G \setminus (H \cup K)$. Then the order of a must be 21. Thus $G = \langle a \rangle$, and so G is cyclic.

14. Let $G = \{(1), (13), (24), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$. For each integer i from 1 to 4, find the stabilizer of i and the orbit of i.

Answer:

i	$\mathrm{Stab}_G i$	orbit of i
1	$(1), (24)$	$\{1, 2, 3, 4\}$
2	$(2), (13)$	$\{1, 2, 3, 4\}$
3	$(1), (24)$	$\{1, 2, 3, 4\}$
4	$(1), (13)$	$\{1, 2, 3, 4\}$