Name: ____________________

1	2	3	4	5	6	7	8	9	10	11	12	Σ

Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Fall 2010**

Final Exam

1. Let G be the group defined by the following Cayley table.

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	5	4	7	6	1	8	3
3	3	8	5	2	7	4	1	6
4	4	3	6	5	8	7	2	1
5	5	6	7	8	1	2	3	4
6	6	1	8	3	2	5	4	7
7	7	4	1	6	3	8	5	2
8	8	7	2	1	4	3	6	5

(a) For each element $a \in G$, find: the order $|a|$; the inverse a^{-1}; and the centralizer $C(a)$.

a	1	2	3	4	5	6	7	8
$\|a\|$								
a^{-1}								
$C(a)$								

(b) What is the center of G?

2. Let G be an abelian group with identity e, and let H be the set of all elements $x \in G$ that satisfy the equation $x^2 = e$. Prove that H is a subgroup of G.

3. Let $G = \langle a \rangle$ be a group generated by an element a of order $|a| = 30$.

(a) Find all elements of G which generate G.

(b) List all the elements in the subgroup $\langle a^6 \rangle$, together with their respective orders.

(c) What are the generators of the subgroup $\langle a^6 \rangle$?

(d) Find an element in G that has order 3. Does this element generate G?

4. (a) Draw the subgroup lattice of $\mathbb{Z}_{24}$.

(b) Make a table with all the elements of $\mathbb{Z}_{24}$, grouped according to their orders; how many elements of each possible order are there?

5. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 7 & 6 & 3 & 1 & 5 \end{bmatrix}$ and $\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 7 & 5 & 1 & 3 & 6 & 2 \end{bmatrix}$, viewed as elements in the symmetric group S_7.

(a) Compute the products

$\beta\alpha =$

$\alpha\beta =$

(b) Compute the inverses

$\alpha^{-1} =$

$\beta^{-1} =$

(c) Compute the conjugate of β by α:

$\alpha\beta\alpha^{-1} =$

(d) Do α and β commute?

6. Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 8 & 6 & 7 & 1 & 5 & 9 & 2 \end{bmatrix}$, viewed as an element in S_9.

(a) Write α as products of disjoint cycles.

(b) Find the order of α.

(c) Write α as a product of transpositions.

(d) Find the parity of α.

7. Let $\mathbb{R}$ be the additive group of real numbers, and let $\mathbb{R}^*$ be the multiplicative group of non-zero real numbers. Consider the map $\phi\colon \mathbb{R} \to \mathbb{R}^*$ given by $\phi(x) = e^x$.

(a) Show that ϕ is an homomorphism from $\mathbb{R}$ to $\mathbb{R}^*$.

(b) What is the kernel of ϕ?

(c) What is the image of ϕ? For each $y \in \mathrm{im}(\phi)$ find an $x \in \mathbb{R}$ such that $\phi(x) = y$?

(d) Is ϕ injective (i.e., one-to-one)?

(e) Is ϕ surjective (i.e., onto)?

(f) Is ϕ an isomorphism?

8. Show that the following pairs of groups are *not* isomorphic. In each case, explain why.

(a) $U(15)$ and $\mathbb{Z}_8$.

(b) A_4 and D_{12}.

(c) S_4 and $D_6 \times \mathbb{Z}_2$.

9. Let S_3 be the group of permutations of the set $\{1, 2, 3\}$. Consider the subgroups $H = \langle (12) \rangle$ and $K = \langle (123) \rangle$.

(a) Write down all the **left** and **right** cosets of H in S_3. Be sure to indicate the elements of each coset.

(b) What is the order of H?

(c) What is the index of H in S_3?

(d) Is H a normal subgroup of S_3?

(e) Write down all the **left** and **right** cosets of K in S_3. Be sure to indicate the elements of each coset.

(f) What is the index of K in S_3?

(g) Is K a normal subgroup of S_3?

10. (a) List all abelian groups (up to isomorphism) of order 100. Write each such group as a direct product of cyclic groups of prime power order.

(b) Let G be an abelian group of order 100. Suppose that G has exactly 3 elements of order 2, and 4 element of order 5. Determine the isomorphism class of G.

11. Let H be set of all 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, with $a, b, d \in \mathbb{Z}_3$ and $ad \neq 0$.

(a) Show that H is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_3)$.

(b) Is H a normal subgroup of $\mathrm{GL}_2(\mathbb{Z}_3)$?

12. Let $\alpha\colon G \to H$ and $\beta\colon H \to K$ be two homomorphisms.

(a) Show that $\beta \circ \alpha\colon G \to K$ is a homomorphism.

(b) Show that $\ker(\alpha)$ is a normal subgroup of $\ker(\beta \circ \alpha)$.

(c) Show that $\mathrm{im}(\beta \circ \alpha)$ is a subgroup of $\mathrm{im}(\beta)$.