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MATH 3175 **Group Theory** **Spring 2024**

Solutions to Homework 4

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1. Let S_4 be the group of permutations of the set $\{1, 2, 3, 4\}$. Consider the subgroup H generated by the cyclic permutation $(1\ 3\ 4) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{smallmatrix}\right)$.

(i) Write down all the right cosets and all the left cosets of H in S_4. (Make sure to indicate all the elements in each coset.)

Right cosets of H in G:

$$\begin{aligned} H &= \{(), (134), (143)\} \\ H(12) &= \{(12), (1342), (1432)\} \\ H(34) &= \{(34), (14), (13)\} \\ H(23) &= \{(23), (1234), (1423)\} \\ H(234) &= \{(234), (14)(23), (123)\} \\ H(132) &= \{(132), (12)(34), (142)\} \\ H(24) &= \{(24), (1324), (1243)\} \\ H(124) &= \{(124), (13)(24), (243)\}. \end{aligned}$$

Left cosets of H in G:

$$\begin{aligned} H &= \{(), (134), (143)\} \\ (12)H &= \{(12), (1234), (1243)\} \\ (34)H &= \{(34), (13), (14)\} \\ (23)H &= \{(23), (1324), (1432)\} \\ (234)H &= \{(234), (13)(24), (142)\} \\ (132)H &= \{(132), (14)(23), (243)\} \\ (24)H &= \{(24), (1342), (1423)\} \\ (124)H &= \{(124), (12)(34), (123)\}. \end{aligned}$$

(ii) What is the index of H in S_4?

$$[G : H] = \#\{\text{right cosets}\} = \#\{\text{left cosets}\} = \frac{|G|}{|H|} = \frac{24}{3} = 8.$$

(iii) Is H a normal subgroup of S_4?

No, H is not a normal subgroup of G. One reason is that the right and left cosets of H and G do not coincide. For instance, $H(12) \neq (12)H$.

Alternatively, it is not the case that $ghg^{-1} \in H$ for all $g \in G$. For instance, taking $g = (12)$ and $h = (134) \in H$, we have $ghg^{-1} = (12)(134)(12) = (234) \notin H$.

2. Let $D_4 = \langle a, b \mid a^4 = b^2 = 1, ba = a^{-1}b\rangle$ be the dihedral group of order 8 (the group of symmetries of the square, with the generator a corresponding to $90°$ clockwise rotation and the generator b corresponding to a reflection in a vertical axis bisecting the square.)

(i) Let $H = \langle a\rangle$ be the cyclic subgroup generated by a. Write down all the right cosets and all the left cosets of H in D_4. Is H a normal subgroup?

Right cosets of H in G:

$$H = \{1, a, a^2, a^3\}$$
$$Hb = \{b, ab, a^2b, a^3b\}$$

Right cosets of H in G:

$$H = \{1, a, a^2, a^3\}$$
$$bH = \{b, ba, ba^2, ba^3\}$$

Yes, H is a normal subgroup of G. Indeed, since $ba = a^3b$, $ba^2 = a^2b$, and $ba^3 = ab$, we have that $bH = Hb$, and of course $H = H$, thus showing that the left and right cosets of H coincide. Alternatively, one may note that $bH = Hb = G \setminus H$, an argument which shows that any index 2 subgroup of an arbitrary group G is normal.

(ii) Let $K = \langle b \rangle$ be the cyclic subgroup generated by b. Write down all the right cosets and all the left cosets of K in D_4. Is K a normal subgroup?

Right cosets of K in G:

$$K = \{1, b\}$$
$$Ka = \{a, ba\}$$
$$Ka^2 = \{a^2, ba^2\}$$
$$Ka^3 = \{a^3, ba^3\}$$

Left cosets of K in G:

$$K = \{1, b\}$$
$$aK = \{a, ab\}$$
$$a^2K = \{a^2, a^2b\}$$
$$a^3K = \{a^3, a^3b\}$$

No, K is not a normal subgroup of G. Indeed, $Ka \neq aK$. Alternatively, $aba^{-1} = a^2b \notin K$, although $b \in K$.

3. Let G be set of all 2×2 matrices in $\mathrm{GL}_2(\mathbb{Z}_3)$ of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, with $a, b, c \in \mathbb{Z}_3$ and $ad \neq 0$.

(i) Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_3)$.

Since $\mathrm{GL}_2(\mathbb{Z}_3)$ is a finite group, it is enough to verify that G is closed under matrix multiplication. Let $A, \bar{A} \in G$; then:

$$A \cdot \bar{A} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{d} \end{pmatrix} = \begin{pmatrix} a\bar{a} & a\bar{b} + b\bar{d} \\ 0 & d\bar{d} \end{pmatrix}$$

and this matrix clearly belongs to G, since $ad \neq 0$ and $\bar{a}\bar{d} \neq 0$ implies $(a\bar{a})(d\bar{d}) \neq 0$.

(ii) Find the order of G.

Since $a, d \in \mathbb{Z}_3^\times$ and $b \in \mathbb{Z}_3$, we have that

$$|G| = \left|\mathbb{Z}_3^\times\right| \cdot \left|\mathbb{Z}_3^\times\right| \cdot |\mathbb{Z}_3| = 2 \cdot 2 \cdot 3 = 12.$$

(iii) Is G a normal subgroup of $\mathrm{GL}_2(\mathbb{Z}_3)$? No, G is not a normal subgroup. For instance, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$ and $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_3)$, but

$$BAB^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix} \notin G.$$

4. Let G be a group. Let $f\colon G \to G$ be the function given by $f(x) = x^{-1}$. Moreover, for each $a \in G$, let $\phi_a\colon G \to G$ be the function given by $\phi_a(x) = axa^{-1}$.

(i) Show that the functions f and ϕ_a are bijections.

The function f has inverse function $f^{-1} = f$; thus, f is a bijection.

The function ϕ_a has inverse function $(\phi_a)^{-1} = \phi_{a^{-1}}$; thus, ϕ_a is a bijection.

(ii) Show that f is an isomorphism if and only if G is abelian.

By part (i), the function f is an isomorphism if and only if it is a homomorphism, that is, $f(ab) = f(a)f(b)$ for all $a, b \in G$, meaning that $(ab)^{-1} = a^{-1}b^{-1}$, for all $a, b \in G$. Inverting both sides, this condition is equivalent to $ab = ba$ for all $a, b \in G$, which means that G is abelian.

(iii) Show that the functions ϕ_a are isomorphisms, for all $a \in G$.

Let $a \in G$. For all $b, c \in G$, properties of group operations give:

$$\phi_a(bc) = a(bc)a^{-1} = (ab)(a^{-1}a)(ca^{-1}) = (aba^{-1})(aca^{-1}) = \phi_a(b)\phi_a(c),$$

showing that ϕ_a is a homomorphism. Therefore, by part (i), ϕ_a is an isomorphism.

5. Let $G = Q_8 \times \mathbb{Z}_2$.

(i) Construct a surjective homomorphism $\varphi\colon G \to \mathbb{Z}_2 \times \mathbb{Z}_2$, and write down $\varphi(x)$ for every $x \in G$.

First, we construct a surjective homomorphism $\psi\colon Q_8 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by defining it on generators by $\psi(i) = (1, 0)$ and $\psi(j) = (0, 1)$ and verifying that the resulting map is indeed a homomorphism. We then have $\psi(k) = \psi(ij) = (1, 1)$, and thus ψ is surjective.

Finally, we extend ψ to a homomorphism $\varphi\colon Q_8 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by setting $\varphi(g, 0) = \varphi(g, 1) = (\psi(g), 0)$. Explicitly, the (surjective) map φ is given by

x	$(1,0)$	$(-1,0)$	$(i,0)$	$(-i,0)$	$(j,0)$	$(-j,0)$	$(k,0)$	$(-k,0)$
$\varphi(x)$	$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$

x	$(1,1)$	$(-1,1)$	$(i,1)$	$(-i,1)$	$(j,1)$	$(-j,1)$	$(k,1)$	$(-k,1)$
$\varphi(x)$	$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$

(ii) Show that there is no surjective homomorphism $\varphi\colon G \to \mathbb{Z}_4 \times \mathbb{Z}_2$.

Let's first show that there is no surjective homomorphism $\psi\colon Q_8 \to \mathbb{Z}_4$. Indeed, if such a homomorphism were to exist, some element of order 4 in Q_8 would have to map to $1 \in \mathbb{Z}_4$. Without loss of generality, we may assume $\psi(i) = 1$. We now analyze the possible values for $\psi(j)$:

- If $\psi(j) = 0$ or 2, then $\psi(-1) = \psi(j^2) = \psi(j) + \psi(j) = 0$.
- If $\psi(j) = 1$ or 3, then $\psi(k) = \psi(ij) = \psi(i) + \psi(j) = 1 + 1 = 2$, and so $\psi(-1) = \psi(k^2) = 2 + 2 = 0$.

In either case, we got $\psi(-1) = 0$. But this contradicts $\psi(-1) = \psi(i^2) = \psi(i) + \psi(i) = 2$.

Alternatively, note that $iji^{-1}j^{-1} = k(-i)(-j) = k^2 = -1$ and $\psi(iji^{-1}j^{-1}) = \psi(i) + \psi(j) - \psi(i) - \psi(j) = 0$. Thus, $\psi(-1) = 0$. It follows that $\psi(i)^2 = \psi(i^2) = \psi(-1) = 0$, and so $\psi(\pm i)$ has order 2. Likewise, $\psi(\pm j)$ and $\psi(\pm k)$ also have order 2, and so the image of ψ does contain $1 \in \mathbb{Z}_4$.

The existence of a surjective homomorphism $\varphi\colon G \to \mathbb{Z}_4 \times \mathbb{Z}_2$ is ruled out by similar arguments. For instance, note that $\varphi(iji^{-1}j^{-1}, x) = (0, 0)$, for $x = 0$ or 1 in $\mathbb{Z}_2$. As above, this implies $\varphi(-1, 0) = \varphi(-1, 1) = (0, 0)$. It follows that $\varphi(i, x)^2 = \varphi(i^2, 2x) = \varphi(-1, 0) = 0$, and so $\varphi(\pm i, 0)$ and $\varphi(\pm i, 1)$ have order 2, and likewise with i replaced by j or k. Therefore, the image of φ consists only of elements of order 1 or 2, and so cannot be equal to $\mathbb{Z}_4 \times \mathbb{Z}_2$, which has elements of order 4.

A more advanced argument goes as follows. Since $\mathbb{Z}_4 \times \mathbb{Z}_2$ is abelian, any homomorphism from G to it must factor through the "abelianization" of G, which is the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. But this group has no elements of order 4, so there is no surjective homomorphism $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_4 \times \mathbb{Z}_2$.