Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Spring 2024**

Homework 4

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1. Let S_4 be the group of permutations of the set $\{1, 2, 3, 4\}$. Consider the subgroup H generated by the cyclic permutation $(1\ 3\ 4) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{smallmatrix}\right)$.

(i) Write down all the right cosets and all the left cosets of H in S_4. (Make sure to indicate all the elements in each coset.)

(ii) What is the index of H in S_4?

(iii) Is H a normal subgroup of S_4?

2. Let $D_4 = \langle a, b \mid a^4 = b^2 = 1, ba = a^{-1}b \rangle$ be the dihedral group of order 8 (the group of symmetries of the square, with the generator a corresponding to $90°$ clockwise rotation and the generator b corresponding to a reflection in a vertical axis bisecting the square.)

(i) Let $H = \langle a \rangle$ be the cyclic subgroup generated by a. Write down all the right cosets and all the left cosets of H in D_4. Is H a normal subgroup?

(ii) Let $K = \langle b \rangle$ be the cyclic subgroup generated by b. Write down all the right cosets and all the left cosets of K in D_4. Is K a normal subgroup?

3. Let G be set of all 2×2 matrices in $\mathrm{GL}_2(\mathbb{Z}_3)$ of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, with $a, b, c \in \mathbb{Z}_3$ and $ad \neq 0$.

(i) Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_3)$.

(ii) Find the order of G.

(iii) Is G a normal subgroup of $\mathrm{GL}_2(\mathbb{Z}_3)$?

4. Let G be a group. Let $f : G \to G$ be the function given by $f(x) = x^{-1}$. Moreover, for each $a \in G$, let $\phi_a : G \to G$ be the function given by $\phi_a(x) = axa^{-1}$.

(i) Show that the functions f and ϕ_a are bijections.

(ii) Show that f is an isomorphism if and only if G is abelian.

(iii) Show that the functions ϕ_a are isomorphisms, for all $a \in G$.

5. Let $G = Q_8 \times \mathbb{Z}_2$.

(i) Construct a surjective homomorphism $\varphi : G \to \mathbb{Z}_2 \times \mathbb{Z}_2$, and write down $\varphi(x)$ for every $x \in G$.

(ii) Show that there is no surjective homomorphism $\varphi : G \to \mathbb{Z}_4 \times \mathbb{Z}_2$.