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MATH 3175 | **Group Theory** | **Spring 2024**

Solutions to Homework 3

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1. Let H and K be two subgroups of a group G.

(i) Is $H \cup K$ a subgroup of G? If yes, give a proof, if no, give a counterexample.

$H \cup K$ need not be a subgroup of G. For instance, take $G = \mathbb{Z}_6$, $H = \langle 2 \rangle = \{0, 2, 4\}$, and $K = \langle 3 \rangle = \{0, 3\}$. Then $H \cup K = \{0, 2, 3, 4\}$ is not a subgroup, since this subset of G is not closed under addition: $2, 3 \in H \cup K$, yet $2 + 3 = 5 \notin H \cup K$.

(ii) Is $H \cap K$ a subgroup of G? If yes, give a proof, if no, give a counterexample.

$H \cap K$ is a subgroup of G. Indeed, let $a, b \in H \cap K$ be two arbitrary elements; then

$$a, b \in H \cap K \Rightarrow \quad a, b \in H \Rightarrow \quad ab^{-1} \in H \quad \text{since } H \text{ is a subgroup}$$
$$a, b \in H \cap K \Rightarrow \quad a, b \in K \Rightarrow \quad ab^{-1} \in K \quad \text{since } K \text{ is a subgroup.}$$

Therefore, $ab^{-1} \in H \cap K$, and tis shows that $H \cap K$ is a subgroup of G.

2. Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8.

(i) Write down the multiplication table of Q_8 and list the orders of the elements in Q_8.

$\cdot$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

g	1	-1	i	$-i$	j	$-j$	k	$-k$
$\mathrm{ord}(g)$	1	2	4	4	4	4	4	4

(ii) Find all the subgroups of Q_8, and determine which ones are cyclic.

There are 6 subgroups of Q_8: $\{1\}$, $\{\pm 1\}$, $\{\pm 1, \pm i\}$, $\{\pm 1, \pm j\}$, $\{\pm 1, \pm k\}$, $\mathbb{Q}_8$. All proper subgroups are cyclic (with generators 1, -1, i, j, k, respectively), but Q_8 itself is not cyclic (the minimum number of generators is 2; for instance, $Q_8 = \langle i, j \rangle$).

(iii) Find all the subgroups of $Q_8 \times \mathbb{Z}_2$, and determine which ones are cyclic.

There are 19 subgroups of $Q_8 \times \mathbb{Z}_2$, with 10 of those being cyclic. Below is the complete list of subgroups (in decreasing size), indicating for each the following data: (1) the isomorphism type, (2) a generating set, (3) the list of elements, and (4) whether the group is cyclic or not.

(1) $Q_8 \times \mathbb{Z}_2 = \langle (i,0), (j,0), (1,1) \rangle = \{(1,0), (-1,0), (i,0), (-i,0), (j,0), (-j,0), (k,0), (-k,0), (1,1), (-1,1), (i,1), (-$ is not cyclic.

(2) $Q_8 \times \{0\} = \langle (i,0), (j,0) \rangle = \{(1,0), (-1,0), (i,0), (-i,0), (j,0), (-j,0), (k,0), (-k,0)\}$ is not cyclic.

(3) $Q_8 = \langle (i,1),(j,1)\rangle = \{(1,0),(i,1),(-1,0),(-i,1),(j,1),(k,0),(-j,1),(-k,0)\}$ is not cyclic.

(4) $Q_8 = \langle (i,1),(k,1)\rangle = \{(1,0),(i,1),(-1,0),(-i,1),(k,1),(-j,0),(-k,1),(k,0)\}$ is not cyclic.

(5) $Q_8 = \langle (j,1),(k,1)\rangle = \{(1,0),(j,1),(-1,0),(-j,1),(k,1),(i,0),(-k,1),(-i,0)\}$ is not cyclic.

(6) $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle (i,0),(1,1)\rangle = \{(1,0),(-1,0),(i,0),(-i,0),(1,1),(-1,1),(i,1),(-i,1)\}$ is not cyclic.

(7) $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle (j,0),(1,1)\rangle = \{(1,0),(-1,0),(j,0),(-j,0),(1,1),(-1,1),(j,1),(-j,1)\}$ is not cyclic.

(8) $\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle (k,0),(1,1)\rangle = \{(1,0),(-1,0),(k,0),(-k,0),(1,1),(-1,1),(k,1),(-k,1)\}$ is not cyclic.

(9) $\mathbb{Z}_4 \times \{0\} = \langle (i,0)\rangle = \{(1,0),(-1,0),(i,0),(-i,0)\}$ is cyclic.

(10) $\mathbb{Z}_4 \times \{0\} = \langle (j,0)\rangle = \{(1,0),(-1,0),(j,0),(-j,0)\}$ is cyclic.

(11) $\mathbb{Z}_4 \times \{0\} = \langle (k,0)\rangle = \{(1,0),(-1,0),(k,0),(-k,0)\}$ is cyclic.

(12) $\mathbb{Z}_4 = \langle (i,1)\rangle = \{(1,0),(-1,0),(i,1),(-i,1)\}$ is cyclic.

(13) $\mathbb{Z}_4 = \langle (j,1)\rangle = \{(1,0),(-1,0),(j,1),(-j,1)\}$ is cyclic.

(14) $\mathbb{Z}_4 = \langle (k,1)\rangle = \{(1,0),(-1,0),(k,1),(-k,1)\}$ is cyclic.

(15) $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (-1,0),(0,1)\rangle = \{(1,0),(-1,0),(0,1),(-1,1)\}$ is not cyclic.

(16) $\mathbb{Z}_2 \times \{0\} = \langle (-1,0)\rangle = \{(1,0),(-1,0)\}$ is cyclic.

(17) $\{1\} \times \mathbb{Z}_2 = \langle (1,1)\rangle = \{(1,0),(1,1)\}$ is cyclic.

(18) $\mathbb{Z}_2 = \langle (-1,1)\rangle = \{(1,0),(-1,1)\}$ is cyclic.

(19) $\{1\} \times \{0\} = \langle (1,0)\rangle = \{(1,0)\}$ is cyclic.

Note: As illustrated in this example, not all subgroups of a direct product of groups are direct products of subgroups. For instance, $\mathbb{Z}_2 = \langle (-1,1)\rangle$ is embedded "diagonally" in the direct product $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle (-1,0),(0,1)\rangle$. From the above list, subgroups (3), (4), (5), (12), (13), (14), and (18) are *not* direct products of subgroups of Q_8 and $\mathbb{Z}_2$, but all other 12 subgroups on the list are direct products of subgroups.

3. Let $\mathbb{Z}_n^\times$ be the multiplicative group of invertible elements in $\mathbb{Z}_n$.

(i) Which of the groups $\mathbb{Z}_6^\times$, $\mathbb{Z}_8^\times$, $\mathbb{Z}_9^\times$, and $\mathbb{Z}_{15}^\times$ are cyclic?

(1) $\mathbb{Z}_6^\times = \{1,5\} = \langle 5\rangle$ is cyclic of order 2.

(2) $\mathbb{Z}_8^\times = \{1,3,5,7\} = \langle 3,5\rangle$ is not cyclic (it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$).

(3) $\mathbb{Z}_9^\times = \{1,2,4,5,7,8\} = \langle 2\rangle$ is cyclic of order 6.

(4) $\mathbb{Z}_{15}^\times = \{1,2,4,7,8,11,13,14\} = \langle 2,14\rangle$ is not cyclic (it is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$).

(ii) Which of the groups $\mathbb{Z}_7^\times$, $\mathbb{Z}_{10}^\times$, $\mathbb{Z}_{12}^\times$, and $\mathbb{Z}_{14}^\times$ are isomorphic?

(1) $\mathbb{Z}_7^\times = \{1,2,3,4,5,6\} = \langle 3\rangle$ is cyclic of order 6.

(2) $\mathbb{Z}_{10}^\times = \{1,3,7,9\} = \langle 3\rangle$ is cyclic of order 4.

(3) $\mathbb{Z}_{12}^\times = \{1,5,7,11\} = \langle 5,7\rangle$ is not cyclic (it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$).

(4) $\mathbb{Z}_{14}^{\times} = \{1, 3, 5, 9, 11, 13\} = \langle 3 \rangle$ is cyclic of order 6.

The groups in this list have orders 6, 4, 4, and 6. Groups of different order cannot be isomorphic (since there is no bijection between them).

Among the remaining two pairs we need to analyze, $\mathbb{Z}_{10}^{\times}$ and $\mathbb{Z}_{12}^{\times}$ are not isomorphic, since the first group is cyclic, whereas the second one is not (alternatively, the first one has elements of order 4 and the second has no such elements).

The groups in the final pair, $\mathbb{Z}_{7}^{\times}$ and $\mathbb{Z}_{14}^{\times}$, *are* isomorphic, since both are cyclic of order 6, and so both are isomorphic to $\mathbb{Z}_6$. An explicit isomorphism $\mathbb{Z}_{7}^{\times} \to \mathbb{Z}_{14}^{\times}$ is given by $[3^k]_7 \mapsto [3^k]_{14}$ for $1 \leqslant k \leqslant 6$, that is,

$$[1]_7 \mapsto [1]_{14},\ [2]_7 \mapsto [9]_{14},\ [3]_7 \mapsto [3]_{14},\ [4]_7 \mapsto [11]_{14},\ [5]_7 \mapsto [5]_{14},\ [6]_7 \mapsto [13]_{14}.$$

4. Let $f\colon G \to H$ be a homomorphism.

(i) Show that $\mathrm{ord}(a) \geqslant \mathrm{ord}(f(a))$, for all $a \in G$.

Recall that the order of an element $a \in G$ is either infinite, or else it equals the positive integer $n \coloneqq \min\{k \in \mathbb{N} : a^k = e_G\}$. Thus, there are two cases to consider.

First suppose $\mathrm{ord}(a) = \infty$. Then either $\mathrm{ord}(f(a)) = \infty$ or $\mathrm{ord}(f(a)) < \infty$; in the first case, $\mathrm{ord}(a) = \mathrm{ord}(f(a)) = \infty$, and in the second case $\mathrm{ord}(a) > \mathrm{ord}(f(a))$. Either way, $\mathrm{ord}(a) \geqslant \mathrm{ord}(f(a))$.

Now suppose $n = \mathrm{ord}(a) < \infty$. Then $a^n = e_G$. Moreover, since f is a homomorphism, $f(a^n) = f(a)^n$. Therefore, $f(a)^n = e_H$. Hence, by the definition of the order of $f(a) \in H$, we must have $\mathrm{ord}(f(a)) \leqslant n$, thus showing that $\mathrm{ord}(a) \geqslant \mathrm{ord}(f(a))$ in this case, too. (In fact, more is true: $\mathrm{ord}(f(a))$ divides $\mathrm{ord}(a)$.)

(ii) Give an example where $\mathrm{ord}(a) > \mathrm{ord}(f(a))$, for some homomorphism $f\colon G \to H$ and some $a \in G$.

Let G be any non-trivial group, let H be any group, and let $f\colon G \to H$ be the trivial homomorphism, given by $f(x) = e_H$ for all $x \in G$. Take an element $a \in G$ with $a \neq e_G$. Then $\mathrm{ord}(a) > 1$ but $\mathrm{ord}(f(a)) = 1$.

(iii) If f is an isomorphism, show that $\mathrm{ord}(a) = \mathrm{ord}(f(a))$, for all $a \in G$.

Let $f^{-1}\colon H \to G$ be the inverse homomorphism. By part (i), we have that $\mathrm{ord}(b) \geqslant \mathrm{ord}(f^{-1}(b))$, for all $b \in H$. Therefore, setting $b = f(a)$, we get that $\mathrm{ord}(f(a)) \geqslant \mathrm{ord}(f^{-1}(f(a))) = \mathrm{ord}(a)$. Using now again part (i), we conclude that $\mathrm{ord}(a) = \mathrm{ord}(f(a))$, for all $a \in G$.

5. Let $\mathbb{Z}_n$ be the cyclic group of order n and let $\mathbb{Z}$ be the (additive) group of integers.

(i) List all the homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_2$.

(1) The trivial homomorphism, $\mathbb{Z}_4 \to \mathbb{Z}_2$, $[k]_4 \to [0]_2$.

(2) The only non-trivial homomorphism, $\mathbb{Z}_4 \to \mathbb{Z}_2$, $[k]_4 \to [k]_2$.

(ii) List all the homomorphisms from $\mathbb{Z}_2$ to $\mathbb{Z}_4$.

(1) The trivial homomorphism, $\mathbb{Z}_2 \to \mathbb{Z}_4$, $[k]_2 \to [0]_4$.

(2) The only non-trivial homomorphism, $\mathbb{Z}_2 \to \mathbb{Z}_4$, $[k]_2 \to [2k]_4$.

(iii) List all the homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}$.

All elements in $\mathbb{Z}_n$ have finite order, whereas all non-zero elements of $\mathbb{Z}$ have infinite order. Therefore, by Problem 4(i), if $f\colon \mathbb{Z}_n \to \mathbb{Z}$ is a homomorphism, we must have $f([k]_n) = 0$ for all $0 \leqslant k < n$. That is, the only homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}$ is the trivial homomorphism.