Prof. Alexandru Suciu

MATH 3175 **Group Theory** **Spring 2024**

Homework 3

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1. Let H and K be two subgroups of a group G.

(i) Is $H \cup K$ a subgroup of G? If yes, give a proof, if no, give a counterexample.

(ii) Is $H \cap K$ a subgroup of G? If yes, give a proof, if no, give a counterexample.

2. Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8.

(i) Write down the multiplication table of Q_8 and list the orders of the elements in Q_8.

(ii) Find all the subgroups of Q_8, and determine which ones are cyclic.

(iii) Find all the subgroups of $Q_8 \times \mathbb{Z}_2$, and determine which ones are cyclic.

3. Let $\mathbb{Z}_n^\times$ be the multiplicative group of invertible elements in $\mathbb{Z}_n$.

(i) Which of the groups $\mathbb{Z}_6^\times$, $\mathbb{Z}_8^\times$, $\mathbb{Z}_9^\times$, and $\mathbb{Z}_{15}^\times$ are cyclic?

(ii) Which of the groups $\mathbb{Z}_7^\times$, $\mathbb{Z}_{10}^\times$, $\mathbb{Z}_{12}^\times$, and $\mathbb{Z}_{14}^\times$ are isomorphic?

4. Let $f\colon G \to H$ be a homomorphism.

(i) Show that $\operatorname{ord}(a) \geq \operatorname{ord}(f(a))$, for all $a \in G$.

(ii) Give an example where $\operatorname{ord}(a) > \operatorname{ord}(f(a))$, for some homomorphism $f\colon G \to H$ and some $a \in G$.

(iii) If f is an isomorphism, show that $\operatorname{ord}(a) = \operatorname{ord}(f(a))$, for all $a \in G$.

5. Let $\mathbb{Z}_n$ be the cyclic group of order n and let $\mathbb{Z}$ be the (additive) group of integers.

(i) List all the homomorphisms from $\mathbb{Z}_4$ to $\mathbb{Z}_2$.

(ii) List all the homomorphisms from $\mathbb{Z}_2$ to $\mathbb{Z}_4$.

(iii) List all the homomorphisms from $\mathbb{Z}_n$ to $\mathbb{Z}$.